Stability in the Stefan problem with surface tension (II)

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Abstract

Continuing our study of the Stefan problem with surface tension effect, in this paper, we establish sharp nonlinear stability and instability of steady circles. Our nonlinear stability proof relies on an energy method along the moving domain, and the discovery of a new 'momentum conservation law'. Our nonlinear instability proof relies on a variational framework which leads to the sharp growth rate estimate for the linearized problem, as well as a bootstrap framework to overcome the nonlinear perturbation with severe high-order derivatives.

Contents

1 Introduction

The Stefan problem is one of the best known parabolic two-phase free boundary problems. It is a simple model of phase transitions in liquid-solid systems.

Let $\Omega\subset\mathbb{R}^2$ denote a domain that contains a liquid and a solid separated by an interface Γ. As the melting or cooling take place the boundary moves and we are naturally led to a free boundary problem. Define the solid phase $\Omega^-(t)$ as a region encircled by $\Gamma(t)$ and define the liquid phase $\Omega^+(t):=\Omega\setminus\overline{\Omega^-}$. The unknowns are the location of the interface $\Big\{\Gamma(t);t\geq 0\Big\}$ and the temperature function $v:[0,T]\times\Omega\to\mathbb{R}$. Let Γ_0 be the initial position of the free boundary and $v_0:\Omega\to\mathbb{R}$ be the initial temperature. We denote the normal velocity of Γ by V and normalize it to be *positive if* Γ *is locally expanding* $\Omega^+(t)$. Furthermore, we denote the mean curvature of Γ by κ. With these notations, (v,Γ) satisfies the following free boundary value problem:

$$\partial_t v-\Delta v=0 \quad \text{in} \quad \Omega\setminus\Gamma. \tag{1.1}$$

$$v=\sigma\kappa \quad \text{on} \quad \Gamma, \tag{1.2}$$

$$V=[v_n]^+_- \quad \text{on} \quad \Gamma, \tag{1.3}$$

$$v_n=0 \quad \text{on} \quad \partial\Omega, \tag{1.4}$$

$$v(0,\cdot)=v_0; \quad \Gamma(0)=\Gamma_0 \tag{1.5}$$

where $\sigma>0$ is the surface tension coefficient. Given v, we write v^+ and v^- for the restriction of v to $\Omega^+(t)$ and $\Omega^-(t)$, respectively. With this notation $[v_n]^+_-$ stands for the jump of the normal derivatives across the interface $\Gamma(t)$, namely $[v_n]^+_-:=v_n^+-v_n^-$, where n stands for the unit normal on the hypersurface $\Gamma(t)$ with respect to $\Omega^-(t)$.

Steady states and the perturbation (u,f)**.** It is well known (cf. [28]) that there are steady circles for the problem (1.1) - (1.5) and they are parametrized by triples $(\bar R,x_0,y_0)$, where $\bar R>0$ and x_0,y_0 are real numbers such that the ball $B_{\bar R}(x_0,y_0)$ with radius $\bar R$ centered at (x_0,y_0) belongs to Ω. To each such triple, we associate a time-independent solution given by $(\bar v,\bar\Gamma)\equiv(\sigma/\bar R,S_{\bar R}(x_0,y_0))$, where $S_{\bar R}(x_0,y_0)=\partial B_{\bar R}(x_0,y_0)$.

Coordinates. We shall assume that $\Omega=B_{R_*}(0)\subset\mathbb{R}^2$ is a ball of radius $R_*>1$. We shall parametrize Γ as a 'graph' over the unit circle $\mathbb{S}^1$ centered at (x_0,y_0) with $\sqrt{x_0^2+y_0^2}+1<R_*$:

$$\Gamma(t)=\big\{x\in\Omega\big|\,x=(R(t,\theta)\cos\theta+x_0,\,R(t,\theta)\sin\theta+y_0)\big\}, \tag{1.6}$$

where $\theta\in\mathbb{S}^1$. Here $R:[0,\infty[\times\mathbb{S}^1\to\mathbb{R}$ is a sufficiently smooth function such that $\bigcup_{0\leq t\leq T}\Gamma(t)\subset\Omega$, $T>0$ and additionally $\Gamma_0=\big\{x\in\Omega\big|\,x=(R_0(\theta)\cos\theta+x_0,R_0(\theta)\sin\theta+y_0)\big\}$ for some $R_0>0$. We introduce a map $\phi:[0,\infty[\times\mathbb{S}^1\to[0,\infty[\times\Gamma$,

$$\phi(t,\theta):=(R(t,\theta)\cos\theta+x_0,R(t,\theta)\sin\theta+y_0). \tag{1.7}$$

The map ϕ induces a metric on Γ, whose line element is denoted by $|g|d\theta$ and in local coordinates $|g|=\sqrt{R^2+R_\theta^2}$. With these notations, we can express the unit tangent τ, the unit normal n, the mean curvature κ and the normal velocity V in local coordinates, i.e. as a function of θ. It follows that

$$\begin{gathered}\tau=\frac{(R_\theta\cos\theta-R\sin\theta,R_\theta\sin\theta+R\cos\theta)}{|g|},\quad n=-\frac{(R_\theta\sin\theta+R\cos\theta,-R_\theta\cos\theta+R\sin\theta)}{|g|},\\ \kappa\circ\phi=H=\frac{1}{|g|}-\frac{1}{R}\Big(\frac{R_\theta}{|g|}\Big)_\theta \quad\text{and}\quad V\circ\phi=\phi_t n=-\frac{R_tR}{|g|}.\end{gathered} \tag{1.8}$$

In the rest of the paper we shall normalize $\sigma=1$ and analyze the stability of the steady state $(\bar v,\bar\Gamma)\equiv(1,S_1(x_0,y_0))$. Hence, it is natural to introduce the perturbation $f:[0,\infty[\times\mathbb{S}^1\to\mathbb{R}$ from the steady circle

$S_1(x_0,y_0)$ by setting $f:=R-1$. Similarly we set $u:=v-1$. Linearizing the curvature H and the Jacobian $|g|$ around 1, we obtain

$$H(f)=1-f-f_{\theta\theta}+N(f); \quad |g|=1+f+\frac{f_\theta^2}{2}+\Psi(f) \tag{1.9}$$

where $N(f)$ denotes the nonlinear remainder in the expansion of $H(f)$ and $\Psi(f)=O(|f|^3+|f_\theta|^3)$. If we set $u_0=v_0-1$ and $f_0=R_0-1$, we can formulate the problem in terms of the perturbation (u,f):

$$\partial_t u-\Delta u=0 \quad \text{in} \quad \Omega\setminus\Gamma. \tag{1.10}$$

$$u=\kappa-1 \quad \text{on} \quad \Gamma, \tag{1.11}$$

$$V=[u_n]_-^+ \quad \text{on} \quad \Gamma, \tag{1.12}$$

$$u_n=0 \quad \text{on} \quad \partial\Omega, \tag{1.13}$$

$$u(0,\cdot)=u_0; \quad f(0)=f_0 \tag{1.14}$$

Motivation. To motivate our results let us first investigate the eigenvalue problem for the linearized Stefan problem around a steady circle $\mathbb{S}^1$ (centered at $(0,0)$):

$$\lambda v-\Delta v=0 \quad \text{on } \Omega\setminus\mathbb{S}^1, \tag{1.15}$$

$$v=-f-f_{\theta\theta} \quad \text{on } \mathbb{S}^1, \tag{1.16}$$

$$[v_n]_-^+=-\lambda f \quad \text{on } \mathbb{S}^1, \tag{1.17}$$

$$v_n=0 \quad \text{on } \partial\Omega;\ [v]_-^+=0 \quad \text{on } \mathbb{S}^1, \tag{1.18}$$

with $\lambda\neq 0$. Observe that RHS of the equation (1.16) simply is the linearization around 1 of the curvature operator H. Note that by integrating (1.15) over $\Omega\setminus\mathbb{S}^1$ and using the boundary condition (1.17), we obtain the first important identity $\lambda\Big(\int_\Omega v+\int_{\mathbb{S}^1} f\Big)=0$. Multiplying (1.15) by v, integrating over $\Omega\setminus\mathbb{S}^1$ and using the boundary conditions (1.16) and (1.17), we are led to the identity

$$\lambda\Big(\int_\Omega v^2+\int_{\mathbb{S}^1}\big\{f_\theta^2-f^2\big\}\Big)+\int_\Omega|\nabla v|^2=0. \tag{1.19}$$

Setting $\mathcal{I}(v,f):=\int_\Omega v^2+\int_{\mathbb{S}^1}\big\{f_\theta^2-f^2\big\}$ and denoting $\mathbf{P}v=v-\frac{1}{|\Omega|}\int_\Omega v$, $\mathbb{P}f=f-\frac{1}{|\mathbb{S}^1|}\int_{\mathbb{S}^1} f$, we note that

$$\mathcal{I}(v,f)=\frac{1}{|\Omega|}\Big(\int_\Omega v\Big)^2+\int_\Omega|\mathbf{P}v|^2+\int_{\mathbb{S}^1}\big\{f_\theta^2-(\mathbb{P}f)^2\big\}-\frac{1}{|\mathbb{S}^1|}\Big(\int_{\mathbb{S}^1} f\Big)^2=\Big(\frac{1}{|\Omega|}-\frac{1}{|\mathbb{S}^1|}\Big)\Big(\int_\Omega v\Big)^2+\tilde{\mathcal{I}}(v,f),$$

where we note that $\tilde{\mathcal{I}}(v,f)=\int_\Omega|\mathbf{P}v|^2+\int_{\mathbb{S}^1}\big\{f_\theta^2-(\mathbb{P}f)^2\big\}\geq 0$ by Wirtinger's inequality and $\int_\Omega v=-\int_{\mathbb{S}^1} f$. Hence, if

$$\zeta:=\frac{1}{|\Omega|}-\frac{1}{|\mathbb{S}^1|}>0,$$

then we easily see that for $\lambda>0$, equation (1.19) can not have non-trivial solutions and thus, all the eigenvalues are non-positive. On the other hand, if $\zeta<0$, one can show that there exists a pair (v,f) such that $\mathcal{I}(v,f)<0$. One can formulate a variational problem so that there exists $\lambda_0>0$

$$-\frac{1}{\lambda_0}=\min\Big\{\frac{\mathcal{I}(v,f)}{\int_\Omega|\nabla v|^2}\Big|(v,f)\in\mathcal{C}\Big\},$$

where the constraint set $\mathcal{C}$ is given by

$$\mathcal{C}:=\Big\{(v,f)\Big|\,v\in H^1(\Omega_{\mathbb{S}^1}),\ f\in H^{5/2}(\mathbb{S}^1),\ v=-f-f_{\theta\theta}\text{ on }\mathbb{S}^1,\ \int_{\mathbb{S}^1} fe^{i\theta}=0,\ \int_\Omega|\nabla v|^2\neq 0\Big\}.$$

The minimizer (v_0,z_0) is the eigenvector associated to λ_0 and leads to the fastest growing mode $e^{\lambda t}(v_0,z_0)$ for the linear Stefan problem. For a complete analysis of the eigenvalue problem via a different method, see [28].

The above analysis suggests that in the *stable case* $\zeta>0$, we should expect the solutions to exist globally close to the steady sphere $\mathbb{S}^1$. Contrary to that, in the *unstable case* $\zeta<0$, an instability should develop.

Notation. The normal velocity of a moving surface Σ is denoted by V_Σ and the index Σ is dropped when it is clear what surface we are referring to. We also set $\Omega_\Sigma:=\Omega\setminus\Sigma$. For a given function $f\in L^2(\mathbb{S}^1)$, we introduce the Fourier decomposition

$$f(\theta)=\sum_{k=0}^{\infty}a_k\cos k\theta+b_k\sin k\theta.$$

For any $n\geq 0$, we define the projection operators $\mathbb{P}_n$ and $\mathbb{P}_{n+}$:

$$\mathbb{P}_n f:=a_n\cos n\theta+b_n\sin n\theta, \quad \mathbb{P}_{n+}f:=\sum_{k=n}^{\infty}a_k\cos k\theta+b_k\sin k\theta. \tag{1.20}$$

For the sake of simplicity, we denote $\mathbb{P}f:=\mathbb{P}_{1+}f=f-\mathbb{P}_0 f$. If a function w is prescribed on Ω then $\mathbf{P}_0 w:=\frac{1}{|\Omega|}\int_\Omega w$ and $\mathbf{P}w:=w-\mathbf{P}_0 w$. Furthermore, we will typically leave out the domain when referring to a Sobolev norm of functions $g:\mathbb{S}^1\to\mathbb{R}$, i.e. $||g||_{H^s}:=||g||_{H^s(\mathbb{S}^1)}$. We shall often employ the following notation: $\partial_b^a g:=\partial_{\theta^a}\partial_{t^b}g$. For a given set $A\subset\mathbb{R}^2$, we denote the closure of A by $\mathrm{cl}(A)$. For any $t\in]0,\infty]$ introduce

$$C^\infty([0,t[;\Omega^\pm):=\Big\{u:[0,t]\times\Omega\to\mathbb{R};\, u\mathbf{1}_{\mathrm{cl}(\Omega^\pm)}\in C^\infty([0,t[\times\mathrm{cl}(\Omega^\pm))\Big\},$$

and analogously define $C^\infty(\Omega^\pm)$ by suppressing the time dependence. For any $k\in\mathbb{N}\cup\{0\}$ set

$$H^k(\Omega^\pm)=\Big\{u:\Omega\to\mathbb{R};\, u\mathbf{1}_{\mathrm{cl}(\Omega^\pm)}\in H^k(\mathrm{cl}(\Omega^\pm))\Big\};\ ||u||_{H^k(\Omega^\pm)}:=||u\mathbf{1}_{\mathrm{cl}(\Omega^+)}||_{H^k(\mathrm{cl}(\Omega^+))}+||u\mathbf{1}_{\mathrm{cl}(\Omega^-)}||_{H^k(\mathrm{cl}(\Omega^-))}, \tag{1.21}$$

where H^k stands for the standard $W^{k,2}$ Sobolev space.

Conservation laws. Assume for the moment u and f are classical solutions to (1.10) - (1.12). The first 'conservation-of-mass' law arises by simply integrating (1.10) over Ω_Γ, using the Stokes' formula and the boundary conditions:

$$\partial_t\int_\Omega u+\partial_t\int_{\mathbb{S}^1}\{f+\frac{f^2}{2}\}=0. \tag{1.22}$$

The second law states the fundamental energy identity; it arises by multiplying (1.1) by $(1+u)$, integrating over Ω and making use of the fact that $\int_\Gamma V\kappa\,dS=\partial_t|\Gamma(t)|$ (where $|\Gamma(t)|=\int_{\mathbb{S}^1}|g|\,d\theta$ stands for the surface volume of $\Gamma(t)$):

$$\frac{1}{2}\partial_t\int_\Omega(1+u)^2+\partial_t\int_{\mathbb{S}^1}\sqrt{(1+f)^2+f_\theta^2}+\int_\Omega|\nabla u|^2=0. \tag{1.23}$$

In addition to these two well-known identities, we discover a new 'conservation-of-momentum' law. It arises by multiplying the equation (1.10) by the harmonic functions $p_a(x,y)=x+\frac{R_*^2 x}{|x|^2}$ and $p_b(x,y)=y+\frac{R_*^2 x}{|y|^2}$ respectively. After integrating over Ω_Γ, using the Green's identity and the boundary conditions, we arrive at:

$$\partial_t\int_\Omega up_a=\partial_t\int_{\mathbb{S}^1}\big\{F_a(R,\theta)-F_a(1,\theta)\big\};\quad \partial_t\int_\Omega up_b=\partial_t\int_{\mathbb{S}^1}\big\{F_b(R,\theta)-F_b(1,\theta)\big\}. \tag{1.24}$$

Here

$$F_a(R,\theta):=\int^R\big(r\cos\theta+x_0+R_*^2\frac{r\cos\theta+x_0}{(r\cos\theta+x_0)^2+(r\sin\theta+y_0)^2}\big)\,dr \tag{1.25}$$

$$F_b(R,\theta):=\int^R\big(r\sin\theta+y_0+R_*^2\frac{r\sin\theta+y_0}{(r\cos\theta+x_0)^2+(r\sin\theta+y_0)^2}\big)\,dr. \tag{1.26}$$

The details of the proof are presented in Lemma 3.9. Let us denote

$$m_0=\int_\Omega u_0+\int_{\mathbb{S}^1}\{f_0+\frac{f_0^2}{2}\}, \tag{1.27}$$

$$m_a:=\int_\Omega u_0p_a-\int_{\mathbb{S}^1}\big\{F_a(R(0),\theta)-F_a(1,\theta)\big\};\quad m_b:=\int_\Omega u_0p_b-\int_{\mathbb{S}^1}\big\{F_b(R(0),\theta)-F_b(1,\theta)\big\}. \tag{1.28}$$

These conservation laws play an important role in our proof of the stability.

1.1 Main theorems in the stable case

Motivated by (1.23), we naturally define the zero-th order energy and dissipation as

$$\mathcal{E}_{(0)}(u,f) := \frac{1}{2}\int_\Omega u^2 + \int_\Omega u + \int_{\mathbb{S}^1}\left(\sqrt{(1+f)^2+f_\theta^2}-1\right); \quad \mathcal{D}_{(0)}(\mathcal{U},\omega) = \int_\Omega |\nabla \mathcal{U}|^2. \tag{1.29}$$

Let $l\in\mathbb{N}$ and $l\geq 3$. By $\mathcal{E}_{(+)}$ we shall denote the energy quantity involving t-derivatives:

$$\mathcal{E}_{(+)}(\mathcal{U},\omega) = \frac{1}{2}\sum_{k=1}^{l-1}\int_\Omega \mathcal{U}_{t^k}^2 + \frac{1}{2}\sum_{k=0}^{l-1}\int_\Omega |\nabla \mathcal{U}_{t^k}|^2 + \frac{1}{2}\sum_{k=1}^{l-1}\int_{\mathbb{S}^1}\left\{\omega_{\theta t^k}^2 - \omega_{t^k}^2\right\}. \tag{1.30}$$

In the same vain, $\mathcal{D}_{(+)}$ is defined through:

$$\mathcal{D}_{(+)}(\mathcal{U},\omega) := \sum_{k=1}^{l}\int_\Omega \mathcal{U}_{t^k}^2 + \sum_{k=1}^{l-1}\int_\Omega |\nabla \mathcal{U}_{t^k}|^2 + \sum_{k=1}^{l}\int_{\mathbb{S}^1}\left\{\omega_{\theta t^k}^2 - \omega_{t^k}^2\right\}. \tag{1.31}$$

Of course, from these definitions it is neither clear that $\mathcal{E}_{(0)}$ defines a positive definite functional, nor it is clear that $\mathcal{E}_{(+)}$ and $\mathcal{D}_{(+)}$ are positive definite, due to the presence of the term $-\int_{\mathbb{S}^1}\omega_{t^k}^2$ in the definitions above. However, under the appropriate smallness assumptions the positive definiteness will become apparent later. Summing (1.29), (1.30) and (1.31), we introduce the *temporal* energy and the *temporal* dissipation:

$$\mathcal{E}(\mathcal{U},\omega) := \mathcal{E}_{(0)}(\mathcal{U},\omega) + \mathcal{E}_{(+)}(\mathcal{U},\omega), \tag{1.32}$$

$$\mathcal{D}(\mathcal{U},\omega) := \mathcal{D}_{(0)}(\mathcal{U},\omega) + \mathcal{D}_{(+)}(\mathcal{U},\omega). \tag{1.33}$$

In addition to the (temporal) energy $\mathcal{E}$ and the (temporal) dissipation $\mathcal{D}$, we introduce the higher order space-time energy quantities $\mathfrak{E}$ and $\mathfrak{D}$ that incorporate mixed space-time derivatives. For any $0<\nu\leq 1$, we set:

$$\mathfrak{E}^\nu(u,f) := \sum_{q=0}^{l-1}\nu^q \mathfrak{E}_q(u,f); \quad \mathfrak{D}^\nu(u,f) := \sum_{q=0}^{l-1}\nu^q\mathfrak{D}_q(u,f), \tag{1.34}$$

where, for $0\leq q\leq l-2$

$$\mathfrak{E}_q(u,f) := \frac{1}{2}||\nabla u_{t^{l-1-q}}||^2_{H^{2q}(\Omega^\pm)} + 2^{2q}\int_{\mathbb{S}^1}\left\{|f_{\theta^{2q+2}t^{l-1-q}}|^2 - |f_{\theta^{2q+1}t^{l-1-q}}|^2\right\} - 2^{2q}\int_{\mathbb{S}^1}|\mathbb{P}_{2+}f_{\theta^{2q+2}t^{l-1-q}}|^2\left(\frac{1}{|g|^{4q+3}}-1\right), \tag{1.35}$$

where $\mathbb{P}_{2+}$ is given in (1.20), and

$$\mathfrak{E}_{l-1}(u,f) := \frac{1}{2}||\nabla u||^2_{H^{2(l-1)}(\Omega^\pm)}. \tag{1.36}$$

Observe that $\mathfrak{E}_q$ is positive definite under the smallness assumption on f. Namely, for any $1\leq q\leq l-2$ set $\rho = \partial^{2q+1}_{l-1-q}f$. Then the integral over $\mathbb{S}^1$ in the definition of $\mathfrak{E}_q$ takes the form

$$\mathfrak{E}_q = 2^{2q}\int_{\mathbb{S}^1}\left\{\rho_\theta^2 - \rho^2 - |\mathbb{P}_{2+}\rho_\theta|^2\left(\frac{1}{|g|^{4q+3}}-1\right)\right\} \geq 2^{2q}\int_{\mathbb{S}^1}\left\{C_W|\mathbb{P}_{2+}\rho_\theta|^2 - |\mathbb{P}_{2+}\rho_\theta|^2\left(\frac{1}{|g|^{4q+3}}-1\right)\right\} \geq 0.$$

The first inequality above simply follows from Wirtinger's inequality, and $||\frac{1}{|g|^{4q+3}}-1||_{L^\infty}\leq C||f||_{W^{1,\infty}}$ is small. Similarly, for any $0\leq q\leq l-1$

$$\mathfrak{D}_q(u,f) := ||\nabla u_{t^{l-1-q}}||^2_{H^{2q+1}(\Omega^\pm)} + 2^{2q-1}\int_{\mathbb{S}^1}\left\{|f_{\theta^{2q+1}t^{l-q}}|^2 - |f_{\theta^{2q}t^{l-q}}|^2\right\} \geq 0, \tag{1.37}$$

by the Wirtinger's inequality. When dealing with the case $\nu=1$, we denote the quantities $\mathfrak{E}^1(u,f)$ and $\mathfrak{D}^1(u,f)$ by $\mathfrak{E}(u,f)$ and $\mathfrak{D}(u,f)$ respectively. Finally, for any $0<\alpha$, $0<\nu\leq 1$ we define the *total energy* and *total dissipation* as:

$$E_{\alpha,\nu} := \mathcal{E} + \alpha\mathfrak{E}^\nu; \quad D_{\alpha,\nu} := \mathcal{D} + \alpha\mathfrak{D}^\nu.$$

Theorem 1.1 *Assume $\zeta>0$. Then there exist $0<\alpha,\nu<1$, a constant C_* and a sufficiently small constant $M^*>0$ such that if*

$$E_{\alpha,\nu}(u_0,f_0)+C_*(m_0^2+m_a^2+m_b^2)\leq M^*,$$

then there exists a unique solution to the Stefan problem (1.10) - (1.14) (u,f), satisfying the global bound

$$E_{\alpha,\nu}(u,f)(t)+C_*(m_0^2+m_a^2+m_b^2)+\frac{1}{2}\int_s^t D_{\alpha,\nu}(u,f)(\tau)\,d\tau\leq E_{\alpha,\nu}(u,f)(s)+C_*(m_0^2+m_a^2+m_b^2),\quad t\geq s\geq 0. \tag{1.38}$$

The second theorem improves the stability to an asymptotic stability statement, and states that the solution converges to a 'nearby' steady sphere:

Theorem 1.2 *Under the assumptions from the previous theorem, there exists $(\bar{x}_0,\bar{y}_0)$ close to (x_0,y_0) and $\bar{R}$ close to 1 such that the solution (v,Γ) to the Stefan problem converges exponentially fast to the steady state $(1/\bar{R},S_{\bar{R}}(\bar{x}_0,\bar{y}_0))$.*

Remark. The analogous statement holds for the steady states $(v,\Gamma)\equiv(\sigma/\bar{R},S_{\bar{R}}(x_0,y_0))$ of the original Stefan problem (1.1) - (1.3). Here, the stability condition reads $\zeta(\sigma,\bar{R}):=\frac{1}{\sigma|\Omega|}-\frac{1}{|S_{\bar{R}}|\bar{R}^2}>0$, where $|S_{\bar{R}}|=2\pi\bar{R}$ is the length of the circle of radius $\bar{R}$.

Remark. The role of the weight ν in the definition (1.34) of $\mathfrak{E}^\nu$ and $\mathfrak{D}^\nu$ is technical in nature. It is introduced naturally in order to close the energy estimates in Subsection 3.2.5.

1.2 Main theorem in the unstable case

As it is shown in Subsection 4.4, with the help of an appropriate change of variables, the Stefan problem can be formulated as a problem on a fixed domain $\Omega_{S_1(x_0,y_0)}$ for the unknowns $w:\Omega_{S_1(x_0,y_0)}\to\mathbb{R}$ and $f:\mathbb{S}^1\to\mathbb{R}$ and it takes the form

$$\partial_t(w,f)=\mathcal{L}(w,f)+U(w,f) \tag{1.39}$$

where the linear operator $\mathcal{L}$ and the nonlinearity U are defined explicitly by (4.400) and (4.424) respectively. In order to formulate the instability statement, we need to specify the notion of the initial perturbation. Lemma 4.3 based on [28], under the instability assumption $\zeta<0$, shows that the linearized operator for the Stefan problem possesses countably many real eigenvalues $\lambda_0,\lambda_1,\lambda_2,\dots$ with finite multiplicity. The leading eigenvalue λ_0 is positive and simple, $\lambda_1=0$ and all the remaining eigenvalues are strictly negative. If $m(i)$ stands for the multiplicity of the eigenvalue λ_i, we associate the eigenvectors $e_{i,k}$ to the eigenvalue λ_i, where $1\leq k\leq m(i)$. The set $\cup_{\substack{i=0\\ i\neq 1}}^{\infty}\cup_{k=1}^{m(i)}\{e_{i,k}\}$ forms an orthonormal basis with respect to the inner product $\langle\cdot,\cdot\rangle_I$ defined by (4.409). We refer to

$$(w_0,f_0)=\sum_{i=0}^{\infty}\sum_{k=1}^{m(i)}c_{i,k}e_{i,k};\quad c_{0,1}\neq 0 \tag{1.40}$$

as the *generic* profile if there exists a non-trivial contribution along the direction of the growing eigenvector (v_0,z_0), i.e. $c_{0,1}\neq 0$. In the following theorem we assert the local existence for small initial data and nonlinear instability of the steady spheres.

Theorem 1.3 *For any smooth generic profile such that $||w_0||^2_{L^2(\Omega)}+||f_0||^2_{L^2}=1$, there exists a sufficiently small number $\theta_0>0$, such that for any $\delta>0$, there exists a unique solution (w^δ,f^δ) to (1.39) with initial conditions $\{(\delta w_0,\delta f_0)\}_\delta$ which on the time interval $[0,T^\delta:=\frac{1}{\lambda_0}\log\frac{\theta_0}{\delta}]$ satisfies*

$$||(w^\delta,f^\delta)(t)-c_{0,1}\delta e^{\lambda_0 t}(v_0,z_0)||_{L^2}\leq C\delta^2e^{2\lambda_0 t}+C\delta,$$

where C is independent of δ. Furthermore, there exists a constant $K_0>0$ independent of δ, such that

$$||w^\delta(T^\delta)||^2_{L^2(\Omega)}+||f^\delta(T^\delta)||^2_{L^2}\geq K_0. \tag{1.41}$$

Remark. Our theorem shows that the nonlinear unstable dynamics is characterized by the growing mode $e^{\lambda_0 t}(v_0,z_0)$ over the time scale $[0,T^\delta]$. If we perturb away from $S_1(0,0)=\mathbb{S}^1$ we show that v_0 is spherically symmetric ((v_0,z_0) is the eigenvector associated to λ_0), hence our instability does NOT contain morphological changes, but only spreading or shrinking along the radial direction.

1.3 Previous work

The Stefan problem has been studied in a variety of mathematical literature over the past century (see for instance [34]). If we set $\sigma=0$ in (1.2), the resulting problem is called the *classical* Stefan problem. It has been known that the classical Stefan problem admits unique global weak solutions in several dimensions ([13], [14] and [23]). The references to the regularity of weak solutions of the two-phase classical Stefan problem are, among others, [2], [3], [4], [9]. Local classical solutions are established in [22] and [27].

If the diffusion equation (1.1) is replaced by the elliptic equation $\Delta u=0$, then the resulting problem is called the quasi-stationary Stefan problem with surface tension (also known as Hele-Shaw or Mullins-Sekerka problem). Global existence for the two-phase quasi-stationary Stefan problem close to a sphere in two dimensions has been obtained in [5] and [8], and in arbitrary dimensions in [12]. Global stability for the one-phase quasi-stationary Stefan problem is established in [16]. Local-in-time solutions in parabolic Hölder spaces in arbitrary dimensions are established in [6].

As to the Stefan problem with surface tension (also known as the Stefan problem with Gibbs-Thomson correction), global weak existence theory (without uniqueness) is developed in [1], [25] and [30]. In [15] the authors consider the Stefan problem with small surface tension i.e. $\sigma\ll 1$ in (1.2). The local existence for the Stefan problem is studied in [29]. In [10] the authors prove a local existence and uniqueness result in suitable Besov spaces, relying on the L^p-regularity theory. Linear stability and instability results for spheres (which are equilibria for the Stefan problem with surface tension) are contained in [28]. Some of the references for the Stefan problem with surface tension and kinetic undercooling effects are [7], [29], [31], [33].

1.4 Strategy of the proofs and basic ideas

In [21], the authors constructed the first global smooth solution near a flat surface. This article is a continuation of such a study. However, there are several major mathematical novelties in our current work. Let us first note that just from the definition (1.29) of $\mathcal{E}_{(0)}$ it is not even clear that this quantity is positive definite. An important step is to Taylor-expand the term $\sqrt{(1+f)^2+f_\theta^2}-1$. We obtain

$$\mathcal{E}_{(0)}=\frac{1}{2}\int_\Omega u^2+\int_\Omega u+\int_{\mathbb{S}^1} f+\frac{1}{2}\int_{\mathbb{S}^1} f_\theta^2+\int_{\mathbb{S}^1} O(|f|^3+|f_\theta|^3)$$

To extract a positive quadratic contribution, we exploit the conservation-of-mass law (1.22) and note that $\int_\Omega u+\int_{\mathbb{S}^1} f=-\frac{1}{2}\int_{\mathbb{S}^1} f^2$. Using this observation, by a calculation similar to the one in the equation following (1.19), we can conclude

$$\mathcal{E}_{(0)}=\frac{1}{2}\int_\Omega |\mathbf{P}u|^2+\frac{1}{2}\zeta\Big(\int_\Omega u\Big)^2+\frac{1}{2}\int_{\mathbb{S}^1}\left\{f_\theta^2-|\mathbb{P}f|^2\right\}+O(|f|^3+|f_\theta|^3)$$

Note that $\int_{\mathbb{S}^1}\left\{f_\theta^2-|\mathbb{P}f|^2\right\}\geq 0$ by the Wirtinger's inequality. Hence, if $\zeta>0$ it does seem that $\mathcal{E}_{(0)}$ is non-negative, modulo the third order remainder. Unfortunately, note that the L^2-norm of the first mode $\mathbb{P}_1 f$ (recall (1.20)) is certainly not controlled by the expression $\int_{\mathbb{S}^1}\left\{f_\theta^2-|\mathbb{P}f|^2\right\}=\int_{\mathbb{S}^1}\left\{|\mathbb{P}_{2+}f_\theta|^2-|\mathbb{P}_{2+}f|^2\right\}$. The analysis of the first modes turns out to be impossible without the new conservation-of-momentum law (1.24). More specifically, the problem lies in ensuring the positivity of $\mathcal{E}_{(0)}$ due to the non-trivial presence of the first modes in the third order remainder. The key to circumvent this severe difficulty is to exploit (1.24), so to write:

$$(1+R_*^2)a_1=\int_\Omega up_a+\int_{\mathbb{S}^1} O(|f|^2+|f_\theta^2|),\ (1+R_*^2)b_1=\int_\Omega up_b+\int_{\mathbb{S}^1} O(|f|^2+|f_\theta^2|);$$

and thus, we can bound the L^2-norm of $\mathbb{P}_1 f$ by the L^2-norms of u and $(\mathbb{P}_0+\mathbb{P}_{2+})f$, which are in turn bounded by the energy $\mathcal{E}_{(0)}$. See Lemma 3.10.

Energy. Unlike our previous work [21], the higher-order energy estimate is considerably more delicate due to the geometric structure of our problem. In order to preserve the boundary condition, one needs to take tangential derivatives. Unfortunately, usual spatial derivatives ∂_{x^i} could lead to linear growth in time when we interchange from one local chart to another, which presents a difficulty in constructing the global solution. Instead, we divide our main energy estimate in two steps. We note that ∂_t is the only globally

defined vector field which is 'almost' tangential with respect to $\Gamma(t)$, so we start by deriving the estimate for the temporal energy, defined as in (1.32) and (1.33):

$$\sup_{0\le s\le t}\mathcal{E}(s)+\int_0^t\mathcal{D}(s)\le\mathcal{E}(0)+C\sup_{0\le s\le t}(\sqrt{\mathcal{E}}+\sqrt{\mathfrak{E}})\int_0^t\mathcal{D}(s)\,ds. \tag{1.42}$$

Instead of deriving the similar energy estimates for the other tangential derivatives, we make use of the temporal energy estimate (1.42) and the heat equation in the bulk to bound $\mathfrak{E}$ by $\mathcal{E}$ and $\mathcal{D}$. We refer to this as the *reduction* step. The estimate we obtain for the space-time energy is the following:

$$\sup_{0\le s\le t}\mathfrak{E}^\nu(s)+\int_0^t\mathfrak{D}^\nu(s)\le\mathfrak{E}^\nu(0)+C\sup_{0\le s\le t}\sqrt{\mathfrak{E}^\nu}\int_0^t\mathfrak{D}^\nu(s)\,ds+C^*\int_0^t\mathcal{D}(s)\,ds. \tag{1.43}$$

Based on the previous two estimates, we are able to obtain the a-priori bound for the solution of the original Stefan problem

$$\sup_{0\le s\le t}E_{\alpha,\nu}(s)+\frac{1}{2}\int_0^t D_{\alpha,\nu}(s)\,ds\le E_{\alpha,\nu}(0),$$

which is the central estimate that will allow us to extend the solution globally. We chose to define our energy on a moving domain, with the aim of preserving the transparency of the geometric structure of the problem. We prove the temporal energy estimates in Subsection 3.2.3 and with the space-time energy estimates in Subsection 3.2.5.

Regularization and the iterative procedure. In order to establish the local existence for the Stefan problem, we set-up an iteration scheme, which generates a sequence of iterates $\{(u^m,f^m)\}_{m\in\mathbb{N}}$. As in [21], such an iteration is well defined, but it breaks the natural energy setting due to the lack of exact cancellations in the presence of cross-terms. We design the elliptic regularization

$$-\frac{f_tR}{|g|}-\epsilon\frac{f_{\theta^4t}}{|g|}=[u_n]_-^+\circ\phi.$$

to overcome this difficulty. For a fixed ϵ, we use it to prove that $\{(u^m,f^m)\}_{m\in\mathbb{N}}$ is a Cauchy sequence in the energy space. However, note that the expression $u^{m+1}-u^m$ does not a-priori make sense, since the iterates u^m and u^{m+1} are defined on different domains Ω^{m-1} and Ω^m, respectively. Thus, we transform the regularized Stefan problem to a problem on a fixed domain by applying a suitable change of variables and then prove the Cauchy property.

Stability for $\zeta>0$. After establishing the existence of the global solution under the positivity assumption on ζ, we further prove the exponential decay of the solution to a nearby steady circle which is uniquely determined via conservation laws (1.22) and (1.24). Just like in [21], we have to prove that the energy $E_{\alpha,\nu}$ is bounded by the dissipation $D_{\alpha,\nu}$. However, the situation is considerably more complicated in comparison to the Stefan problem close to flat steady surfaces. The reason is that the Dirichlet condition (1.2) can not be directly used to estimate the L^2-norm of f against the L^2-norm of ∇u, since $\int_\Gamma(\kappa-1)\neq0$. Again, we crucially exploit the conservation laws (1.22) and (1.24) together with the condition $\zeta>0$, to prove this estimate. This is carried out in Subsection 3.4.

Instability for $\zeta<0$. The passage from the linear to nonlinear instability is a delicate problem in general. In Section 4 we start by proving the a-priori bounds and the local existence under the assumption $\zeta<0$ (Lemma 4.1 and Theorem 4.2), analogously to the stable case. However, to prove the instability, we resort to a bootstrap technique as developed by Guo and Strauss in [19]. The main idea is to show that on the time scale of instability, some stronger norm $|||\cdot|||$ (given by the high-order energy) is actually controlled by the weaker norm $||\cdot||$ (which is simply L^2-norm in our case). The details of this general procedure are described in Lemma 4.10. However, in order to apply Lemma 4.10, a detailed understanding of the linear problem and the growing eigenmode is necessary. In Subsection 4.3, we provide a variational characterization of the positive eigenvalue (Lemma 4.5) and establish sharp growth estimate for the linear problem in suitable norms (Lemma 4.9). To finally prove Theorem 1.3 (Subsection 4.4), we have to reformulate the Stefan problem so that the Duhamel principle can be applied. Again, we transform the problem to a problem on the fixed domain with the severe nonlinearity $N(f)$ coming from the curvature term, contained in the

Dirichlet boundary condition. To ensure the linearity of the boundary condition, we further modify the unknowns and as a result obtain the problem (4.420) - (4.423), to which we apply Lemma 4.10.

This work is a continuation of the program of developing a robust energy method to investigate and characterize morphological stabilities/instabilities in the setting of free boundary problems in PDE. In particular, in a forthcoming paper [20], the nonlinear stability and instability of steady spheres in arbitrary dimensions will be analyzed.

Due to the rotational invariance, in the rest of the paper we shall, without loss of generality, analyze the stability of the steady states with $y_0=0$, i.e. of the form $(\bar{v},\bar{\Gamma})=(1,S_1(x_0,0))$.

2 Energy identities

2.1 Regularization

In order to prove Theorem 1.1, we shall regularize the Stefan problem by adding viscosity to the 'jump' relation (1.3). Namely, for $\epsilon>0$, we consider the following equation:

$$V+\epsilon\Lambda=[u_n]_-^+ \quad \text{on } \Gamma, \tag{2.44}$$

where $\Lambda\circ\phi=-\frac{f_{\theta^4 t}}{|g|}$. Thus, expressed as an equation on $\mathbb{S}^1$, the equation (2.44) reads

$$-\frac{f_t R}{|g|}-\epsilon\frac{f_{\theta^4 t}}{|g|}=[u_n]_-^+\circ\phi,$$

where ϕ is given by (1.7) ($y_0=0$). We shall refer to the problem (1.1), (1.2) and (2.44) as the *regularized Stefan problem*. In the rest of this subsection we define a number of energy quantities that will be necessary for our subsequent energy estimates.

The stable case. We introduce the additional ϵ-dependent energy terms that incorporate the viscosity:

$$\mathcal{E}_\epsilon(\mathcal{U},\omega):=\mathcal{E}(\mathcal{U},\omega)+\hat{\mathcal{E}}^\epsilon(\omega); \quad \hat{\mathcal{E}}^\epsilon(\omega):=\frac{1}{2}\sum_{k=0}^{l-1}\epsilon\int_{\mathbb{S}^1}\left\{\omega_{\theta^3 t^k}^2-\omega_{\theta^2 t^k}^2\right\}, \tag{2.45}$$

$$\mathcal{D}_\epsilon(\mathcal{U},\omega):=\mathcal{D}(\mathcal{U},\omega)+\hat{\mathcal{D}}^\epsilon(\omega); \quad \hat{\mathcal{D}}^\epsilon(\omega):=\sum_{k=0}^{l-1}\epsilon\int_{\mathbb{S}^1}\left\{\omega_{\theta^3 t^{k+1}}^2-\omega_{\theta^2 t^{k+1}}^2\right\}. \tag{2.46}$$

Accordingly, with regard to the higher-order space-time norms $\mathfrak{E}$ and $\mathfrak{D}$, for any $0\le q\le l-1$ we define:

$$\hat{\mathfrak{E}}_q^\epsilon(f):=\epsilon 2^{2q}\int_{\mathbb{S}^1}\left\{|f_{\theta^{2q+4}t^{l-1-q}}|^2-|f_{\theta^{2q+3}t^{l-1-q}}|^2\right\}-\epsilon 2^{2q}\int_{\mathbb{S}^1}|\mathbb{P}_{2+}f_{\theta^{2q+4}t^{l-1-q}}|^2\Big(\frac{1}{R|g|^{4q+3}}-1\Big), \tag{2.47}$$

$$\hat{\mathfrak{D}}_q^\epsilon:=\epsilon 2^{2q-1}\int_{\mathbb{S}^1}\left\{|f_{\theta^{2q+3}t^{l-q}}|^2-|f_{\theta^{2q+2}t^{l-q}}|^2\right\} \tag{2.48}$$

We define the total higher order energies in the natural way:

$$\hat{\mathfrak{E}}^{\epsilon,\nu}=\sum_{q=0}^{l-1}\nu^q\hat{\mathfrak{E}}_q^\epsilon;\ \mathfrak{E}_\epsilon^\nu=\mathfrak{E}^\nu+\hat{\mathfrak{E}}^{\epsilon,\nu}; \quad \hat{\mathfrak{D}}^{\epsilon,\nu}=\sum_{q=0}^{l-1}\nu^q\hat{\mathfrak{D}}_q^\epsilon;\ \mathfrak{D}_\epsilon^\nu=\mathfrak{D}^\nu+\hat{\mathfrak{D}}^{\epsilon,\nu}. \tag{2.49}$$

Again, if $\nu=1$, we simply drop the index 1 in the notation introduced in (2.49).

The unstable case. In order to prove the local existence theorem in the case $\zeta<0$, we introduce the appropriate energy quantities. Using (1.20), we define the energy $\mathcal{M}$ by

$$\mathcal{M}(\mathcal{U},\omega):=\frac{1}{2}\sum_{k=0}^{l-1}\int_\Omega\mathcal{U}_{t^k}^2+\frac{1}{2}\sum_{k=0}^{l-1}\int_\Omega|\nabla\mathcal{U}_{t^k}|^2+\frac{1}{2}\sum_{k=0}^{l-1}\int_{\mathbb{S}^1}\left\{\omega_{\theta t^k}^2-|\mathbb{P}_{2+}\omega_{t^k}|^2\right\}; \tag{2.50}$$

and the dissipation $\mathcal{N}$ is defined through:

$$\mathcal{N}(\mathcal{U},\omega) := \sum_{k=1}^{l}\int_\Omega \mathcal{U}_{t^k}^2 + \sum_{k=0}^{l-1}\int_\Omega |\nabla \mathcal{U}_{t^k}|^2 + \sum_{k=1}^{l}\int_{\mathbb{S}^1} \left\{\omega_{\theta t^k}^2 - |\mathbb{P}_{2+}\omega_{t^k}|^2\right\}. \tag{2.51}$$

The quantities $\mathcal{M}$ and $\mathcal{N}$ are the analogues of $\mathcal{E}$ and $\mathcal{D}$ defined by (1.32) and (1.33). In analogy to (1.34), for any $0<\nu\le 1$ we define the space-time energy quantities:

$$\mathfrak{M}^\nu = \sum_{q=0}^{l-1}\nu^q \mathfrak{M}_q, \quad \mathfrak{N}^\nu = \sum_{q=0}^{l-1}\nu^q \mathfrak{N}_q \tag{2.52}$$

where, for any $0\le q\le l-1$, we define

$$\begin{aligned} \mathfrak{M}_q(u,f) &:= ||\nabla u_{t^{l-1-q}}||^2_{H^{2q}(\Omega)} + 2^{2q}\int_{\mathbb{S}^1}\left\{|f_{\theta^{2q+2}t^{l-1-q}}|^2 - |\mathbb{P}_{2+}f_{\theta^{2q+1}t^{l-1-q}}|^2\right\}; \\ \mathfrak{N}_q(u,f) &:= ||\nabla u_{t^{l-1-q}}||^2_{H^{2q+1}(\Omega)} + 2^{2q}\int_{\mathbb{S}^1}\left\{|f_{\theta^{2q+1}t^{l-q}}|^2 - |\mathbb{P}_{2+}f_{\theta^{2q}t^{l-q}}|^2\right\}. \end{aligned} \tag{2.53}$$

Finally, for any $0<\beta$, $0<\nu\le 1$, the *total energy* and the *total dissipation* for the unstable case are defined by

$$M_{\beta,\nu} := \mathcal{M} + \beta\mathfrak{M}^\nu; \quad N_{\beta,\nu} := \mathcal{N} + \beta\mathfrak{N}^\nu.$$

As to the energy quantities that arise from the regularization (2.44), analogously to (2.45) - (2.49), we define

$$\mathcal{M}_\epsilon(\mathcal{U},\omega) := \mathcal{M}(\mathcal{U},\omega) + \hat{\mathcal{M}}^\epsilon(\omega); \ \hat{\mathcal{M}}^\epsilon(\omega) := \sum_{k=0}^{l-1}\epsilon\int_{\mathbb{S}^1}\left\{\omega_{\theta^3 t^k}^2 - |\mathbb{P}_{2+}\omega_{\theta^2 t^k}|^2\right\} \tag{2.54}$$

and

$$\mathcal{N}_\epsilon(\mathcal{U},\omega) := \mathcal{N}(\mathcal{U},\omega) + \hat{\mathcal{N}}^\epsilon(\omega); \ \hat{\mathcal{N}}^\epsilon(\omega) := \sum_{k=0}^{l-1}\epsilon\int_{\mathbb{S}^1}\left\{\omega_{\theta^3 t^{k+1}}^2 - |\mathbb{P}_{2+}\omega_{\theta^2 t^{k+1}}|^2\right\}. \tag{2.55}$$

Furthermore, for any $0\le q\le l-1$ we define:

$$\hat{\mathfrak{M}}_q^\epsilon(f) := \epsilon 2^{2q}\int_{\mathbb{S}^1}\left\{|f_{\theta^{2q+4}t^{l-1-q}}|^2 - |\mathbb{P}_{2+}f_{\theta^{2q+3}t^{l-1-q}}|^2\right\}; \ \hat{\mathfrak{N}}_q^\epsilon(f) := \epsilon 2^{2q}\int_{\mathbb{S}^1}\left\{|f_{\theta^{2q+3}t^{l-q}}|^2 - |\mathbb{P}_{2+}f_{\theta^{2q+2}t^{l-q}}|^2\right\}, \tag{2.56}$$

and hence the total higher order energies are given by

$$\hat{\mathfrak{M}}^{\epsilon,\nu} = \sum_{q=0}^{l-1}\nu^q\hat{\mathfrak{M}}_q^\epsilon; \ \mathfrak{M}_\epsilon^\nu = \mathfrak{M}^\nu + \hat{\mathfrak{M}}^{\epsilon,\nu}; \quad \hat{\mathfrak{N}}^{\epsilon,\nu} = \sum_{q=0}^{l-1}\nu^q\hat{\mathfrak{N}}_q^\epsilon; \ \mathfrak{N}_\epsilon^\nu = \mathfrak{N}^\nu + \hat{\mathfrak{N}}^{\epsilon,\nu}. \tag{2.57}$$

2.2 Trace

Let γ stand for the trace operator on the curve Γ. By ∂_t^* we shall denote the time differentiation operator along the hypersurface Γ defined by

$$\partial_t^* u\circ\phi = \partial_t(u\circ\phi). \tag{2.58}$$

With these notations, applying the chain rule in the above formula, we arrive at the following decomposition:

$$\partial_t^* u = \gamma[u_t] + V u_n + V_{\|} u_s, \tag{2.59}$$

where $V_{\|}\circ\phi = f_t(\cos\theta,\sin\theta)\cdot\tau = \frac{f_t f_\theta}{|g|}$ is the tangential velocity, V stands for the normal velocity of the curve $\mathcal{C}$ and $u_s = \partial_s u$ is the tangential derivative of u (recall (1.8)). Note that the pull-back ∂_s^* of the operator ∂_s, with respect to the map $\phi:\mathbb{S}^1\to\Gamma$, is given by $\partial_s^* = \frac{1}{|g|}\partial_\theta$ (in short, in local coordinates $\partial_s = \frac{1}{|g|}\partial_\theta$). Iterating the formula (2.59) k times and using the product rule, we arrive at the following decomposition formula for the operator ∂_{t^k}:

$$\gamma[u_{t^k}] = \partial_{t^k}^* u - \left[\gamma,\partial_{t^k}\right](u), \tag{2.60}$$

where the commutator $\left[\gamma,\partial_{t^k}\right](u)$ is defined through

$$\left[\gamma,\partial_{t^k}\right](u):=\sum_{p=0}^{k-1}\sum_{q+r=k-p} C^{p,q,r}\partial^*_{t^{k-1-p}}(V^qV_{\|}^r)\partial_s^r\partial_n^q u_{t^p}, \tag{2.61}$$

and $C^{p,q,r}$ are some positive real constants. The following trace lemma will be of crucial importance for estimating terms appearing in (2.61):

Lemma 2.1 *Let $v\in H^{k+1}(\Omega)$, $f\in H^k(\mathbb{S}^1)$ and $q,r\in\mathbb{N}$ such that $r+q=k$. Then the following inequality holds:*

$$||\partial_s^r\partial_n^q v||_{L^2(\Gamma)}\leq C\big(1+||f||_{H^k}\big)||v||_{H^{k+1}(\Omega)}.$$

Proof. Using the formulas $\partial_n=n_i\partial_{x^i}$ and $\partial_s=\tau_i\partial_{x^i}$, we may write

$$\partial_s^r\partial_n^q v=\sum_{j=1}^{r}F_j(n)Q_{k+1-j}(v).$$

Here $F_j(n)$ is a function involving at most $j-1$ ∂_s^*-derivatives of n and thus at most j ∂_θ-derivatives of f, since $n=1+O(f_\theta)$. On the other hand $Q_m(u)$ consists of terms containing m Cartesian derivatives of u. The idea is to use the Sobolev estimates: if $j\leq\frac{k+1}{2}$, we estimate $||F_j||_{L^\infty}\leq C||F_j||_{H^1}$ by the Sobolev inequality, whereas Q_{k+j-1} is estimated in L^2-norm. We then use the trace inequality, to conclude $||Q_{k+j-1}||_{L^2}\leq C||u||_{H^{k+j}(\Omega)}$. In the case $j>\frac{k+1}{2}$, we estimate F_j in L^2-norm and Q_{k+j-1} in L^∞-norm together with the Sobolev and trace inequalities. Combining the two cases the claim of the lemma easily follows. Note that we only use the trace inequality for the Cartesian derivatives on Γ. Observe that in the inequality $||v||_{L^2(\Gamma)}\leq C||v||_{H^1(\Omega)}$, the constant C depends only on the C^1-norm of f. However, for $k\geq 2$, the C^1-norm of f is bounded by the H^{k+1}-norm of f. □

2.3 Identities

Let $I=[0,q]$ for some $0<q\leq\infty$. The derivation of the energy identities crucially depends on the following model problem:

$$\mathcal{U}_t-\Delta\mathcal{U}=\mathcal{F}\quad\text{on } \Omega\times I \tag{2.62}$$

$$\mathcal{U}^{\pm}=\bar{\mathcal{U}}+\tilde{\mathcal{U}}^{\pm}\quad\text{on } \Gamma, \tag{2.63}$$

$$\bar{\mathcal{U}}\circ\phi=-\frac{\chi}{F}-\frac{\chi_{\theta^2}}{\psi|g|F}+G(\psi,\chi)\quad\text{on } \mathbb{S}^1, \tag{2.64}$$

$$[\partial_\nu\mathcal{U}]^+_-=\bar{\mathcal{V}}+\tilde{\mathcal{V}}\quad\text{on } \Gamma, \tag{2.65}$$

$$\bar{\mathcal{V}}\circ\phi=-\omega_t\frac{\psi}{|g|F}-\epsilon\frac{\omega_{\theta^4t}}{|g|F}+h(\psi,\omega)\quad\text{on } \mathbb{S}^1. \tag{2.66}$$

We should think of $\tilde{\mathcal{U}}^{\pm}$, $\tilde{\mathcal{V}}$, $G(\psi,\chi)$ and $h(\psi,\chi)$ as certain kind of 'error terms' that will not contribute to the extraction of the energy. We first define the stable case energies $\bar{\mathcal{E}}_\epsilon$ and $\bar{\mathcal{D}}_\epsilon$ (for the model problem) through:

$$\bar{\mathcal{E}}_\epsilon(\mathcal{U},\omega):=\frac{1}{2}\int_\Omega\Big\{\mathcal{U}^2+|\nabla\mathcal{U}|^2\Big\}+\frac{1}{2}\int_{\mathbb{S}^1}\Big\{\omega_\theta^2-\omega^2\Big\}+\frac{1}{2}\epsilon\int_{\mathbb{S}^1}\Big\{\omega_{\theta^3}^2-\omega_{\theta^2}^2\Big\} \tag{2.67}$$

and

$$\bar{\mathcal{D}}_\epsilon(\mathcal{U},\omega):=\int_\Omega\Big\{\mathcal{U}_t^2+|\nabla\mathcal{U}|^2\Big\}+\int_{\mathbb{S}^1}\Big\{\omega_{\theta t}^2-\omega_t^2\Big\}+\epsilon\int_{\mathbb{S}^1}\Big\{\omega_{\theta^3t}^2-\omega_{\theta^2t}^2\Big\}. \tag{2.68}$$

As to the unstable case, we define the energies $\bar{\mathcal{M}}_\epsilon$ and $\bar{\mathcal{N}}_\epsilon$ by

$$\bar{\mathcal{M}}_\epsilon(\mathcal{U},\omega):=\frac{1}{2}\int_\Omega\Big\{\mathcal{U}^2+|\nabla\mathcal{U}|^2\Big\}+\frac{1}{2}\int_{\mathbb{S}^1}\big\{\omega_\theta^2-|\mathbb{P}_{2+}\omega|^2\big\}+\frac{1}{2}\epsilon\int_{\mathbb{S}^1}\big\{\omega_{\theta^3}^2-|\mathbb{P}_{2+}\omega_{\theta^2}|^2\big\}. \tag{2.69}$$

$$\bar{\mathcal{N}}_\epsilon(\mathcal{U},\omega):=\int_\Omega\Big\{\mathcal{U}_t^2+|\nabla\mathcal{U}|^2\Big\}+\int_{\mathbb{S}^1}\big\{\omega_{\theta t}^2-|\mathbb{P}_{2+}\omega_t|^2\big\}+\epsilon\int_{\mathbb{S}^1}\big\{\omega_{\theta^3t}^2-|\mathbb{P}_{2+}\omega_{\theta^2t}|^2\big\}. \tag{2.70}$$

Remark. Note that the first modes of ω are included in the energy quantities $\bar{\mathcal{M}}_\epsilon$ and $\bar{\mathcal{N}}_\epsilon$.

Lemma 2.2 *The following identities hold:*

(a)

$$\bar{\mathcal{E}}_\epsilon(t)+\int_0^t \bar{\mathcal{D}}_\epsilon(s)\,ds=\bar{\mathcal{E}}_\epsilon(0)+\int_\Omega(\mathcal{U}+\mathcal{U}_t)\mathcal{F}+\int_0^t\int_\Gamma\Big\{O+P\Big\}+\int_0^t\int_{\mathbb{S}^1}\Big\{Q+S\Big\}, \tag{2.71}$$

(b)

$$\bar{\mathcal{M}}_\epsilon(t)+\int_0^t \bar{\mathcal{N}}_\epsilon(s)\,ds=\bar{\mathcal{M}}_\epsilon(0)+\int_\Omega(\mathcal{U}+\mathcal{U}_t)\mathcal{F}+\int_0^t\int_\Gamma\Big\{O^u+P^u\Big\}+\int_0^t\int_{\mathbb{S}^1}\Big\{Q^u+S^u\Big\}, \tag{2.72}$$

where (recalling (1.8))

$$O=O^u=O(\mathcal{U}):=\tilde{\mathcal{U}}^\pm\mathcal{U}_n^\pm+\bar{\mathcal{U}}\tilde{\mathcal{V}}+V(\mathcal{U}^\pm)^2, \tag{2.73}$$

$$P=P^u=P(\mathcal{U}):=\left(\partial_t^*\tilde{\mathcal{U}}^\pm-[\gamma,\partial_t]\mathcal{U}^\pm\right)\mathcal{U}_n^\pm+\partial_t^*\bar{\mathcal{U}}\tilde{\mathcal{V}}+V|\nabla\mathcal{U}^\pm|^2, \tag{2.74}$$

$$\begin{aligned}
&Q=Q(\chi,\omega,\psi,F)=-\omega_t(\omega-\chi)+\omega_{\theta t}(\omega_\theta-\chi_\theta)-\epsilon\omega_{\theta^2 t}(\omega_{\theta^2}-\chi_{\theta^2})+\epsilon\omega_{\theta^3 t}(\omega_{\theta^3}-\chi_{\theta^3})\\
&+\chi\omega_t(\frac{\psi}{F^2}-1)+\chi_{\theta^2}\omega_t(\frac{1}{|g|F^2}-1)+\epsilon\chi\omega_{\theta^4 t}(\frac{1}{F^2}-1)+\epsilon\chi_{\theta^2}\omega_{\theta^4 t}(\frac{1}{\psi|g|F^2}-1)\\
&+h\bar{\mathcal{U}}\circ\phi|g|-G\big(\omega_t\frac{\psi}{|g|F}+\epsilon\frac{\omega_{\theta^4 t}}{|g|F}\big)|g|;\ Q^u=Q+(\mathbb{P}_0+\mathbb{P}_1)\omega(\mathbb{P}_0+\mathbb{P}_1)\omega_t+\epsilon\mathbb{P}_1\omega_{\theta^2}\mathbb{P}_1\omega_{\theta^2 t}.
\end{aligned} \tag{2.75}$$

$$\begin{aligned}
&S=S(\chi,\omega,\psi,F):=-\omega_t(\omega_t-\chi_t)+\omega_{\theta t}(\omega_{\theta t}-\chi_{\theta t})-\epsilon\omega_{\theta^2 t}(\omega_{\theta^2 t}-\chi_{\theta^2 t})+\epsilon\omega_{\theta^3 t}(\omega_{\theta^3 t}-\chi_{\theta^3 t})\\
&+\chi_t\omega_t(\frac{\psi}{F^2}-1)+\chi_{\theta^2 t}\omega_t(\frac{1}{|g|F^2}-1)+\epsilon\chi_t\omega_{\theta^4 t}(\frac{1}{F^2}-1)+\epsilon\chi_{\theta^2 t}\omega_{\theta^4 t}(\frac{1}{\psi|g|F^2}-1)\\
&-\big(-\chi(F^{-1})_t-\chi_{\theta^2}\big(\frac{1}{\psi|g|F}\big)_t+G_t\big)\bar{\mathcal{V}}\circ\phi|g|+\big(\frac{\chi_t}{F^2}+\frac{\chi_{\theta^2 t}}{\psi|g|F^2}\big)h|g|;\ S^u=S+|(\mathbb{P}_0+\mathbb{P}_1)\omega_t|^2+\epsilon|\mathbb{P}_1\omega_{\theta^2 t}|^2.
\end{aligned} \tag{2.76}$$

Proof. Clearly, part (b) follows easily from part (a). Hence, in the rest of the proof we only focus on the identity in the part (a) of Lemma 2.2. The first energy identity arises by multiplying (2.62) by $\mathcal{U}$ and integrating over Ω. We obtain:

$$\frac{1}{2}\partial_t\int_\Omega\mathcal{U}^2-\int_\Gamma V(\mathcal{U}^\pm)^2+\int_\Omega|\nabla\mathcal{U}|^2-\int_\Gamma\mathcal{U}^\pm\mathcal{U}_n^\pm=\int_\Omega\mathcal{U}\mathcal{F}.$$

Using (2.63) and (2.65):

$$-\int_\Gamma\mathcal{U}^\pm\mathcal{U}_n^\pm=-\int_\Gamma\bar{\mathcal{U}}\bar{\mathcal{V}}-\int_\Gamma\left(\tilde{\mathcal{U}}^\pm\mathcal{U}_n^\pm+\bar{\mathcal{U}}\tilde{\mathcal{V}}\right) \tag{2.77}$$

We shall evaluate $-\int_\Gamma\bar{\mathcal{U}}\bar{\mathcal{V}}$ by transforming it to the integral over $\mathbb{S}^{n-1}$ using (2.64) and (2.66):

$$-\int_\Gamma\bar{\mathcal{U}}\bar{\mathcal{V}}=-\int_{\mathbb{S}^1}\big(\frac{\chi}{F}+\frac{\chi_{\theta^2}}{\psi|g|F}\big)\big(\omega_t\frac{\psi}{|g|F}+\epsilon\frac{\omega_{\theta^4 t}}{|g|F}\big)|g|-\int_{\mathbb{S}^1}h\bar{\mathcal{U}}\circ\phi|g|-\int_{\mathbb{S}^1}G\big(\omega_t\frac{\psi}{|g|F}+\epsilon\frac{\omega_{\theta^4 t}}{|g|F}\big)|g|. \tag{2.78}$$

We proceed toward the extraction of the energy contribution from the first integral on RHS of (2.78). Note that

$$\begin{aligned}
&-\int_{\mathbb{S}^1}\big(\frac{\chi}{F}+\frac{\chi_{\theta^2}}{\psi|g|F}\big)\big(\omega_t\frac{\psi}{|g|F}+\epsilon\frac{\omega_{\theta^4 t}}{|g|F}\big)|g|\,d\theta=-\int_{\mathbb{S}^1}\chi\omega_t\frac{\psi}{F^2}-\int_{\mathbb{S}^1}\frac{\chi_{\theta^2}\omega_t}{|g|F^2}-\epsilon\int_{\mathbb{S}^1}\frac{\chi\omega_{\theta^4 t}}{F^2}-\epsilon\int_{\mathbb{S}^1}\frac{\chi_{\theta^2}\omega_{\theta^4 t}}{\psi|g|F^2}\\
&=\frac{1}{2}\partial_t\int_{\mathbb{S}^1}\left\{\omega_\theta^2-\omega^2\right\}+\frac{1}{2}\epsilon\partial_t\int_{\mathbb{S}^1}\left\{\omega_{\theta^3}^2-\omega_{\theta^2}^2\right\}\\
&+\int_{\mathbb{S}^1}\omega_t(\omega-\chi)-\int_{\mathbb{S}^1}\omega_{\theta t}(\omega_\theta-\chi_\theta)+\epsilon\int_{\mathbb{S}^1}\omega_{\theta^2 t}(\omega_{\theta^2}-\chi_{\theta^2})-\epsilon\int_{\mathbb{S}^1}\omega_{\theta^3 t}(\omega_{\theta^3}-\chi_{\theta^3})\\
&-\int_{\mathbb{S}^1}\chi\omega_t(\frac{\psi}{F^2}-1)-\int_{\mathbb{S}^1}\chi_{\theta^2}\omega_t(\frac{1}{|g|F^2}-1)-\epsilon\int_{\mathbb{S}^1}\chi\omega_{\theta^4 t}(\frac{1}{F^2}-1)-\epsilon\int_{\mathbb{S}^1}\chi_{\theta^2}\omega_{\theta^4 t}(\frac{1}{\psi|g|F^2}-1)
\end{aligned} \tag{2.79}$$

The identities (2.77) - (2.79) give the formulas (2.73) and (2.75) for O and Q, respectively. The second energy identity arises by multiplying (2.62) by $\mathcal{U}_t$ and integrating over Ω. We obtain:

$$\int_\Omega\mathcal{U}_t^2+\frac{1}{2}\partial_t\int_\Omega|\nabla\mathcal{U}|^2-\int_\Gamma V|\nabla\mathcal{U}^\pm|^2-\int_\Gamma\mathcal{U}_t^\pm\mathcal{U}_n^\pm=\int_\Omega\mathcal{U}_t\mathcal{F}.$$

Using (2.63) and (2.65)

$$-\int_\Gamma \mathcal{U}_t^\pm \mathcal{U}_n^\pm = -\int_\Gamma (\partial_t^*\bar{\mathcal{U}} + \partial_t^*\tilde{\mathcal{U}}^\pm - [\gamma,\partial_t]\mathcal{U}^\pm)\mathcal{U}_n^\pm = -\int_\Gamma \partial_t^*\bar{\mathcal{U}}\bar{\mathcal{V}} - \int_\Gamma \Big\{ (\partial_t^*\tilde{\mathcal{U}}^\pm - [\gamma,\partial_t]\mathcal{U}^\pm)\mathcal{U}_n^\pm + \partial_t^*\bar{\mathcal{U}}\tilde{\mathcal{V}}. \Big\} \quad (2.80)$$

We shall evaluate $-\int_\Gamma \partial_t^*\bar{\mathcal{U}}\bar{\mathcal{V}}$ by transforming it to the integral over $\mathbb{S}^1$ using (2.64) and (2.66). By (2.64)

$$\partial_t(\bar{\mathcal{U}}\circ\phi) = -\frac{\chi_t}{F} - \frac{\chi_{\theta^2 t}}{\psi|g|F} - \chi(\frac{1}{F})_t - \chi_{\theta^2}\big(\frac{1}{\psi|g|F}\big)_t + G_t$$

Thus, we have

$$\begin{aligned}
&-\int_\Gamma \partial_t^*\bar{\mathcal{U}}\bar{\mathcal{V}} = -\int_{\mathbb{S}^1} \partial_t(\bar{\mathcal{U}}\circ\phi)\bar{\mathcal{V}}\circ\phi|g| = -\int_{\mathbb{S}^1} \big(-\frac{\chi_t}{F} - \frac{\chi_{\theta^2 t}}{\psi|g|F} - \chi(\frac{1}{F})_t - \chi_{\theta^2}\big(\frac{1}{\psi|g|F}\big)_t + G_t\big)\big(-\frac{\omega_t\psi}{|g|F} - \frac{\omega_{\theta^4 t}}{|g|F} + h\big)|g| \\
&= -\int_{\mathbb{S}^1} \chi_t\omega_t \frac{\psi}{F^2} - \int_{\mathbb{S}^1} \frac{\chi_{\theta^2 t}\omega_t}{|g|F^2} - \epsilon\int_{\mathbb{S}^1} \frac{\chi_t\omega_{\theta^4 t}}{|g|F^2} - \epsilon\int_{\mathbb{S}^1} \frac{\chi_{\theta^2 t}\omega_{\theta^4 t}}{\psi|g|F^2} + \int_{\mathbb{S}^1} \big(-\chi(\frac{1}{F})_t - \chi_{\theta^2}\big(\frac{1}{\psi|g|F}\big)_t + G_t\big)\bar{\mathcal{V}}\circ\phi|g| \\
&-\int_{\mathbb{S}^1} \big(\frac{\chi_t}{F^2} + \frac{\chi_{\theta^2 t}}{\psi|g|F^2}\big)h|g| = \int_{\mathbb{S}^1} \{\omega_{\theta t}^2 - \omega_t^2\} + \epsilon\int_{\mathbb{S}^1} \{\omega_{\theta^3 t}^2 - \omega_{\theta^2 t}^2\} \\
&+\int_{\mathbb{S}^1} \omega_t(\omega_t - \chi_t) - \int_{\mathbb{S}^1} \omega_{\theta t}(\omega_{\theta t} - \chi_{\theta t}) + \epsilon\int_{\mathbb{S}^1} \omega_{\theta^2 t}(\omega_{\theta^2 t} - \chi_{\theta^2 t}) - \epsilon\int_{\mathbb{S}^1} \omega_{\theta^3 t}(\omega_{\theta^3 t} - \chi_{\theta^3 t}) \\
&-\int_{\mathbb{S}^1} \chi_t\omega_t(\frac{\psi}{F^2} - 1) - \int_{\mathbb{S}^1} \chi_{\theta^2 t}\omega_t(\frac{1}{|g|F^2} - 1) - \epsilon\int_{\mathbb{S}^1} \chi_t\omega_{\theta^4 t}(\frac{1}{F^2} - 1) - \epsilon\int_{\mathbb{S}^1} \chi_{\theta^2 t}\omega_{\theta^4 t}(\frac{1}{\psi|g|F^2} - 1) \\
&+\int_{\mathbb{S}^1} \big(-\chi(\frac{1}{F})_t - \chi_{\theta^2}\big(\frac{1}{\psi|g|F}\big)_t + G_t\big)\bar{\mathcal{V}}\circ\phi|g| - \int_{\mathbb{S}^1} \big(\frac{\chi_t}{F^2} + \frac{\chi_{\theta^2 t}}{\psi|g|F^2}\big)h|g|
\end{aligned} \quad (2.81)$$

The identities (2.80) and (2.81) imply the formulas (2.74) and (2.76) for P and S respectively.

3 Stable case $\zeta > 0$

3.1 Iteration scheme

In this section we set up an iteration scheme for proving the local existence of solutions to the Stefan problem (1.10) - (1.14). We want to find a sequence of iterates $(u^k,\Gamma^k)_{k\in\mathbb{N}}$, where each curve Γ^k is expressed as a graph over $\mathbb{S}^1$ via the mapping $\phi^k : [0,\infty[\times\mathbb{S}^1 \to \Gamma^k$, $\phi^k(t,\theta) = (R^k(t,\theta)\cos\theta + x_0, R^k(t,\theta)\sin\theta)$.

The iteration is defined as follows: given (u^m,R^m) for some $m\in\mathbb{N}$ and smooth initial data $u^{m+1}(0,\cdot) = u_0(\cdot)$, find u^{m+1} by solving the equation

$$\partial_t u^{m+1} - \Delta u^{m+1} = 0, \quad \text{on} \quad \Omega_{\Gamma^m}, \quad (3.82)$$

with the given Dirichlet boundary condition

$$u^{m+1} = \kappa^m \quad \text{on} \quad \Gamma^m. \quad (3.83)$$

After we have obtained u^{m+1}, for given smooth initial data $R^{m+1}(0,\cdot) = R_0(\cdot)$, find R^{m+1} by solving the following equation on Γ^m:

$$V^{m+1} + \epsilon\Lambda^{m+1} = [\partial_n u^{m+1}]_-^+, \quad (3.84)$$

where $V^{m+1} : \Gamma^m \to \mathbb{R}$ is given by $V^{m+1}\circ\phi^m = -\frac{f_t^{m+1}R^m}{|g^m|}$ and $\Lambda^{m+1} : \Gamma^m \to \mathbb{R}$ is defined through $\Lambda^{m+1}\circ\phi^m = -\frac{f_{\theta^4 t}^{m+1}}{|g^m|}$. Here $|g^m| = \sqrt{(R^m)^2 + (R_\theta^m)^2}$. The relation (3.84) can be expressed as an equation on $\mathbb{S}^1$:

$$-\frac{f_t^{m+1}R^m}{|g^m|} - \epsilon\frac{f_{\theta^4 t}^{m+1}}{|g^m|} = [\partial_n u^{m+1}]_-^+\circ\phi^m \quad \text{on} \quad \mathbb{S}^1, \quad (3.85)$$

Our goal is to prove that the sequence of iterates (u^m, f^m) converges to the solution of the regularized Stefan problem. Applying the temporal differentiation operator ∂_{t^k} $(0 \le k \le l-1)$ to the equations (3.82), (3.83) and (3.85), we obtain

$$u_{t^{k+1}}^{m+1} - \Delta u_{t^k}^{m+1} = 0 \quad \text{on} \quad \Omega^m, \quad (3.86)$$

$$u^{m+1}_{t^k} = \bar{u}^m_k + \tilde{u}^m_k \quad \text{on} \quad \Gamma^m, \tag{3.87}$$

where

$$\bar{u}^m_k = \partial^*_{t^k}\kappa^m \tag{3.88}$$

and thus, (3.88) can be rewritten as

$$\bar{u}^m_k \circ \phi^m = -f^m_{t^k} - \frac{f^m_{\theta^2 t^k}}{R^m|g^m|} + G^m_k \quad \text{on} \quad \mathbb{S}^1, \tag{3.89}$$

$$[\partial_n u^{m+1}_{t^k}]^+_- = \bar{v}^m_k + \tilde{v}^m_k \quad \text{on} \quad \Gamma^m, \tag{3.90}$$

where

$$\bar{v}^m_k = \partial^*_{t^k}(V^{m+1} + \epsilon\Lambda^{m+1}) \tag{3.91}$$

We can thus rewrite

$$\bar{v}^m_k \circ \phi^m = -\frac{f^{m+1}_{t^{k+1}}R^m}{|g^m|} - \epsilon\frac{f^{m+1}_{\theta^4 t^{k+1}}}{|g^m|} + h^m_k \quad \text{on} \quad \mathbb{S}^1, \tag{3.92}$$

where

$$\tilde{u}^m_k = -[\gamma^m, \partial_{t^k}]u^{m+1}, \tag{3.93}$$

$$\tilde{v}^m_k = -[\gamma^m, \partial_{t^k}][\partial_n u^{m+1}]^+_-, \tag{3.94}$$

$$G^m_k = -\sum_{q=0}^{k-1} f^m_{\theta^2 t^q}\big(\frac{1}{R^m|g^m|}\big)_{t^{k-q}} + \partial_{t^k}N^*(f^m). \tag{3.95}$$

$$h^m_k = -h^m_{k,1} - \epsilon h^m_{k,2}; \quad h^m_{k,1} = -\sum_{q=0}^{k-1} f^{m+1}_{t^{q+1}}\partial_{t^{k-q}}\big(\frac{R^m}{|g^m|}\big),\ h^m_{k,2} = -\sum_{q=0}^{k-1} f^{m+1}_{\theta^4 t^{q+1}}\big(\frac{1}{|g^m|}\big)_{t^{k-q}}. \tag{3.96}$$

In (3.95) above, $N^*(f)$ is the quadratic remainder in the decomposition $H(f) = 1 - f - \frac{f_{\theta\theta}}{R|g|} + N^*(f)$, i.e. $N^*(f) = \frac{1}{|g|} - (1-f) - (\frac{1}{|g|})_\theta \frac{f_\theta}{R}$. As a first step toward the introduction of the energy norm for the iterates, we state the version of the conservation law (1.22) which takes into account the iteration.

Lemma 3.1 *The following conservation law holds:*

$$\int_\Omega u^{m+1}_t + \int_{\mathbb{S}^1} f^{m+1}_t = -\int_{\mathbb{S}^1} f^m f^{m+1}_t, \tag{3.97}$$

Proof. Upon integrating (3.82) over Ω^m and using integration by parts we arrive at

$$\int_\Omega u^{m+1}_t = -\int_{\Gamma^m}[\partial_n u^{m+1}]^+_- = -\int_{\mathbb{S}^1}\Big(\frac{R^m f^{m+1}_t}{|g^m|} + \epsilon\frac{f^{m+1}_{\theta^4 t}}{|g^m|}\Big)|g^m|\,d\theta = -\int_{\mathbb{S}^1} f^{m+1}_t - \int_{\mathbb{S}^1} f^m f^{m+1}_t,$$

which implies the claim. □

From (3.97), we immediately see

$$\int_\Omega u^{m+1}_t + \partial_t\int_{\mathbb{S}^1}\{f^{m+1} + \frac{1}{2}(f^{m+1})^2\} = \int_{\mathbb{S}^1}(f^{m+1} - f^m)f^{m+1}_t. \tag{3.98}$$

For any $k \in \mathbb{N}$ define:

$$\mathcal{E}^k_{(+)} := \mathcal{E}_{\epsilon,+}(u^k, f^k), \qquad \mathcal{D}^k_{(+)} := \mathcal{D}_{\epsilon,+}(u^k, f^k), \tag{3.99}$$

where

$$\mathcal{E}_{\epsilon,+}(\mathcal{U},\omega) = \mathcal{E}_{(+)}(\mathcal{U},\omega) + \frac{1}{2}\sum_{k=1}^{l-1}\epsilon\int_{\mathbb{S}^1}\{\omega^2_{\theta^3 t^k} - \omega^2_{\theta^2 t^k}\};\ \mathcal{D}_{\epsilon,+}(\mathcal{U},\omega) = \mathcal{D}_{(+)}(\mathcal{U},\omega) + \sum_{k=1}^{l-1}\epsilon\int_{\mathbb{S}^1}\{\omega^2_{\theta^3 t^{k+1}} - \omega^2_{\theta^2 t^{k+1}}\}.$$

Setting

$$\mathcal{U}=\partial_{t^k}u^{m+1},\ \omega=R^{m+1}_{t^k},\ \chi=R^m_{t^k},\ \psi=R^m,\ \bar{\mathcal{U}}=\bar{u}^m_k,\ \tilde{\mathcal{U}}=\tilde{u}^m_k,\ \bar{\mathcal{V}}=\bar{V}^m_k,\ \tilde{\mathcal{V}}=\tilde{v}^m_k,\ G=G^m_k,\ h=h^m_k,\ F\equiv 1\ ,\ \mathcal{F}\equiv 0 \tag{3.100}$$

the identity (2.71) implies:

$$\mathcal{E}^{m+1}_{(+)}(t)+\int_0^t\mathcal{D}^{m+1}(s)\,ds=\mathcal{E}_{\epsilon,+}(0)+\int_0^t\int_{\Gamma^m}\Big\{O^m+P^m\Big\}+\int_0^t\int_{\mathbb{S}^1}\Big\{Q^m+S^m\Big\}. \tag{3.101}$$

Here $O^m:=\sum_{k=1}^{l-1}O^m_k$, $P^m=\sum_{k=0}^{l-1}P^m_k$, $Q^m=\sum_{k=1}^{l-1}Q^m_k$ and $S^m=\sum_{k=0}^{l-1}S^m_k$, whereby

$$O^m_k:=O(\partial_{t^k}u^{m+1}),\qquad P^m_k:=P(\partial_{t^k}u^{m+1}) \tag{3.102}$$

and

$$Q^m_k:=Q(R^m_{t^k},R^{m+1}_{t^k},R^m),\qquad S^m_k:=S(R^m_{t^k},R^{m+1}_{t^k},R^m). \tag{3.103}$$

On the level of iterates, for any $j\in\mathbb{N}$ we define the 0-th order energy by

$$\mathcal{E}^j_{(0)}=\frac{1}{2}\int_{\Omega^{j-1}}(\mathbf{P}u^j)^2+\frac{\zeta}{2}\Big(\int_{\Omega^{j-1}}u^j\Big)^2+\frac{1}{2}\int_{\mathbb{S}^1}\big\{(f^j_\theta)^2-|\mathbb{P}f^j|^2\big\}+\frac{1}{2}\epsilon\int_{\mathbb{S}^1}\Big\{|f^j_{\theta^3}|^2-|f^j_{\theta^2}|^2\Big\},$$

$$\mathcal{D}^j_{(0)}=\int_{\Omega^{j-1}}|\nabla u^j|^2.$$

Remark. Note that $\mathcal{E}^j$ is slightly different from $\mathcal{E}_{(0)}$.

3.1.1 The 0-th order energy identity

To derive the 0-th order energy identity we multiply the equation (3.82) by u^{m+1} and integrate over Ω^m. We obtain

$$\frac{1}{2}\partial_t\int_{\Omega^m}(u^{m+1})^2+\int_{\Omega^m}u^{m+1}_t+\int_{\mathbb{S}^1}H^m f^{m+1}_t R^m-\epsilon\int_{\Gamma^m}\kappa^m\Lambda^{m+1}+\int_{\Omega^m}|\nabla u^{m+1}|^2=0. \tag{3.104}$$

We apply the ϵ-dependent part of the formulas (2.78) and (2.79) to evaluate $-\epsilon\int_{\Gamma^m}\kappa^m\Lambda^{m+1}$. With $\chi=f^m$, $\omega=f^{m+1}$, $F\equiv 1$, $\psi=R^m$, $g=g^m$, $G=N^*(f^m)$ and $h\equiv 0$, we obtain

$$-\epsilon\int_{\Gamma^m}\kappa^m\Lambda^{m+1}=-\epsilon\int_{\mathbb{S}^1}H^m\frac{f^{m+1}_{\theta^4 t}}{|g^m|}|g^m|=\frac{1}{2}\epsilon\int_{\mathbb{S}^1}\Big\{|f^k_{\theta^3}|^2-|f^k_{\theta^2}|^2\Big\}-\int_{\mathbb{S}^1}A^m, \tag{3.105}$$

where

$$A^m=-\epsilon f^{m+1}_{\theta^2 t}(f^{m+1}_{\theta^2}-f^m_{\theta^2})+\epsilon f^{m+1}_{\theta^3 t}(f^{m+1}_{\theta^3}-f^m_{\theta^3})+\epsilon f^m_{\theta^2}f^{m+1}_{\theta^4 t}(\frac{1}{R^m|g^m|}-1)+\epsilon N^*(f^m)f^{m+1}_{\theta^4 t}. \tag{3.106}$$

where the quadratic remainder term $N^*(f)$ is defined in the line after (3.96). Using (3.98) and (3.105), we obtain from (3.104):

$$\begin{aligned}&\frac{1}{2}\partial_t\Big\{\int_{\Omega^m}(u^{m+1})^2-\int_{\mathbb{S}^1}(f^{m+1})^2\Big\}-\int_{\mathbb{S}^1}f^{m+1}_t+\frac{1}{2}\partial_t\epsilon\int_{\mathbb{S}^1}\Big\{|f^k_{\theta^3}|^2-|f^k_{\theta^2}|^2\Big\}+\int_{\Omega^m}|\nabla u^{m+1}|^2\\&=-\int_{\mathbb{S}^1}H^m f^{m+1}_t R^m+\int_{\mathbb{S}^1}(f^m-f^{m+1})f^{m+1}_t+\int_{\mathbb{S}^1}A^m.\end{aligned} \tag{3.107}$$

We now make the fundamental calculation:

$$\begin{aligned}&\int_{\Omega^m}(u^{m+1})^2-\int_{\mathbb{S}^1}(f^{m+1})^2=\int_{\Omega^m}(\mathbf{P}u^{m+1})^2\,dx+|\Omega^m|(\mathbf{P}_0u^{m+1})^2-\int_{\mathbb{S}^1}(\mathbb{P}f^{m+1})^2-|\mathbb{S}^1|(\mathbb{P}_0f^{m+1})^2\\&=\int_{\Omega^m}(\mathbf{P}u^{m+1})^2\,dx-\int_{\mathbb{S}^1}(\mathbb{P}f^{m+1})^2+\big(\frac{1}{|\Omega^m|}-\frac{1}{|\mathbb{S}^1|}\big)\big(\int_{\Omega^m}u^{m+1}\big)^2+\frac{1}{|\mathbb{S}^1|}\Big(\big(\int_{\Omega^m}u^{m+1}\big)^2-\big(\int_{\mathbb{S}^1}f^{m+1}\big)^2\Big).\end{aligned} \tag{3.108}$$

Using (3.108) and adding $\frac{1}{2}\partial_t\int_{\mathbb{S}^1}(f_\theta^{m+1})^2$ to both sides of (3.107), we arrive at

$$\partial_t\mathcal{E}_0^{m+1}+\mathcal{D}_{(0)}^{m+1}=A(u^{m+1},f^m,f^{m+1}), \tag{3.109}$$

where

$$\begin{aligned}&A(u^{m+1},f^m,f^{m+1}):=-\frac{1}{2|\mathbb{S}^1|}\partial_t\Big\{\big(\int_{\Omega^m}u^{m+1}\big)^2-\big(\int_{\mathbb{S}^1}f^{m+1}\big)^2\Big\}\\&+\int_{\mathbb{S}^1}\Big\{f_t^{m+1}-H^mf_t^{m+1}R^m+(f^m-f^{m+1})f_t^{m+1}+f_\theta^{m+1}f_{\theta t}^{m+1}\Big\}+\int_{\mathbb{S}^1}A^m.\end{aligned} \tag{3.110}$$

We expand

$$H^mR^m=1-f_{\theta\theta}^m+N_1(f^m),$$

where $N_1(g)=O(|g|^2+|g_\theta|^2+|g_{\theta\theta}|^2)$. We can conveniently rewrite

$$\begin{aligned}&\int_{\mathbb{S}^1}\Big\{f_t^{m+1}-H^mf_t^{m+1}R^m+(f^m-f^{m+1})f_t^{m+1}+f_\theta^{m+1}f_{\theta t}^{m+1}\Big\}\\&=\int_{\mathbb{S}^1}\Big\{f_t^{m+1}-f_t^{m+1}\Big(1-f_{\theta\theta}^m+N_1(f^m)\Big)+(f^m-f^{m+1})f_t^{m+1}+f_\theta^{m+1}f_{\theta t}^{m+1}\Big\}\\&=\int_{\mathbb{S}^1}\Big\{f_t^{m+1}f_{\theta\theta}^m+f_\theta^{m+1}f_{\theta t}^{m+1}+(f^m-f^{m+1})f_t^{m+1}-f_t^{m+1}N_1(f^m)\Big\}\\&=\int_{\mathbb{S}^1}\Big\{\big(f^m-f^{m+1}+f_{\theta\theta}^m-f_{\theta\theta}^{m+1}\big)f_t^{m+1}-f_t^{m+1}N_1(f^m)\Big\}.\end{aligned}$$

Using the previous calculation we can rewrite $A(u^{m+1},f^m,f^{m+1})$:

$$\begin{aligned}&A(u^{m+1},f^m,f^{m+1})=\int_{\mathbb{S}^1}\Big\{\big(f^m-f^{m+1}+f_{\theta\theta}^m-f_{\theta\theta}^{m+1}\big)f_t^{m+1}-f_t^{m+1}N_1(f^m)\Big\}\\&-\frac{1}{2|\mathbb{S}^1|}\partial_t\Big\{\big(\int_{\Omega^m}u^{m+1}\big)^2-\big(\int_{\mathbb{S}^1}f^{m+1}\big)^2\Big\}+\int_{\mathbb{S}^1}A^m.\end{aligned} \tag{3.111}$$

Upon integrating (3.109) in time and using (3.111), we obtain:

$$\mathcal{E}_{(0)}^{m+1}(t)+\int_0^t\mathcal{D}_{(0)}^{m+1}(s)\,ds=\mathcal{E}_{(0)}(0)+\int_0^tA(u^{m+1},f^m,f^{m+1}). \tag{3.112}$$

Summing (3.101) and (3.112), we arrive at the fundamental energy identity:

$$\mathcal{E}_\epsilon^{m+1}(t)+\int_0^t\mathcal{D}_\epsilon^{m+1}(\tau)\,d\tau=\mathcal{E}_\epsilon(0)+\int_0^tA(u^{m+1},f^m,f^{m+1})+\int_0^t\int_{\Gamma^m}\Big\{O^m+P^m\Big\}+\int_0^t\int_{\mathbb{S}^1}\Big\{Q^m+S^m\Big\}. \tag{3.113}$$

3.2 Energy estimates

In this section we prove the basic a-priori estimates for the regularized Stefan problem. In order to bound the higher-order space-time energy, we first introduce the necessary notations. We denote $\mathfrak{E}_q^{j,\nu}:=\mathfrak{E}_q^{\nu}(u^j,f^j)$, $\hat{\mathfrak{E}}_q^{j,\nu}:=\hat{\mathfrak{E}}_q^{\epsilon,\nu}(u^j,f^j)$ and $\mathfrak{E}_{\epsilon,q}^j:=\mathfrak{E}_q^j+\hat{\mathfrak{E}}_q^j$ for any $j\in\mathbb{N}$, where in the definitions (2.53) and (1.36) Ω is substituted by Ω^j. Analogously we define $\mathfrak{D}_q^{j,\nu}$, $\hat{\mathfrak{D}}_q^{j,\nu}$ and $\mathfrak{D}_{\epsilon,q}^j$. For any $j\in\mathbb{N}$ we finally set $\mathfrak{E}^{j,\nu}=\sum_{q=0}^{l-1}\nu^q\mathfrak{E}_q^j$ and we define $\hat{\mathfrak{E}}^{j,\nu}$ analogously. Summing the ϵ-independent energy $\mathfrak{E}^j$ and the ϵ-dependent energy $\hat{\mathfrak{E}}_q^j$, we define $\mathfrak{E}_\epsilon^{j,\nu}:=\mathfrak{E}^{j,\nu}+\hat{\mathfrak{E}}^{j,\nu}$. In a fully analogous fashion, we introduce $\mathfrak{D}^{j,\nu}$, $\hat{\mathfrak{D}}^{j,\nu}$ and $\mathfrak{D}_\epsilon^{j,\nu}$. We follow the notational convention of dropping the superscript ν when $\nu=1$. Let us set

$$E_{\alpha,\nu}^j=\mathcal{E}_\epsilon^{j,\nu}+\alpha\mathfrak{E}_\epsilon^{j,\nu};\quad D_{\alpha,\nu}^j=\mathcal{D}_\epsilon^{j,\nu}+\alpha\mathfrak{D}_\epsilon^{j,\nu}.$$

Lemma 3.2 *Let $\zeta := \frac{1}{|\Omega|} - \frac{1}{|\mathbb{S}^1|} > 0$. Then there exist positive constants α, ν, C_a, C_A, θ_0 and t^ϵ, such that for any $\theta, \theta_1 < \theta_0$, $t \le t^\epsilon$ such that if $E_{\alpha,\nu}(0) < \theta/2$ and*

$$\sum_{k=1}^{l-1} ||\mathbb{P}_0 f^m_{t^k}||_{L^2} + \sum_{k=1}^{l-1} ||\mathbb{P}_1 f^m_{t^k}||^2_{L^2} \le C_a \mathcal{E}^m, \quad \sum_{k=1}^{l} ||\mathbb{P}_0 f^m_{t^k}||_{L^2} + \sum_{k=1}^{l} ||\mathbb{P}_1 f^m_{t^k}||^2_{L^2} \le C_a \mathcal{D}^m,$$

$$E^m_{\alpha,\nu}(t) + \int_0^t D^m_{\alpha,\nu}(\tau)\,d\tau \le \theta, \quad \sup_{0\le s\le t} ||f^m||^2_{H^{2l}} \le C_A(\theta_1 + \int_0^t \mathfrak{D}^{m,\nu})$$

then

$$\sum_{k=1}^{l-1} ||\mathbb{P}_0 f^{m+1}_{t^k}||_{L^2} + \sum_{k=1}^{l-1} ||\mathbb{P}_1 f^{m+1}_{t^k}||^2_{L^2} \le C_a \mathcal{E}^{m+1}, \quad \sum_{k=1}^{l} ||\mathbb{P}_0 f^{m+1}_{t^k}||_{L^2} + \sum_{k=1}^{l} ||\mathbb{P}_1 f^{m+1}_{t^k}||^2_{L^2} \le C_a \mathcal{D}^{m+1}, \quad (3.114)$$

$$E^{m+1}_{\alpha,\nu}(t) + \int_0^t D^{m+1}_{\alpha,\nu}(\tau)\,d\tau \le \theta \quad \text{and} \quad \sup_{0\le s\le t} ||f^{m+1}||^2_{H^{2l}} \le C_A(\theta_1 + \frac{t}{\epsilon^2}\int_0^t \mathfrak{D}^{m+1,\nu}). \quad (3.115)$$

Remark. For any $j\in\mathbb{N}$, any Sobolev exponent $k\in\mathbb{N}\cup\{0\}$ and a function $v:\Omega^j\to\mathbb{R}$, we shall denote the Sobolev norm $||v||_{H^k(\Omega^{j,\pm})}$ simply by $||v||_{H^k(\Omega^j)}$ (recall that $||\cdot||_{H^k(\Omega^\pm)}$ is defined in (1.21)).

3.2.1 Proof of (3.114) (zero-th and the first modes) and second claim of (3.115)

Let $p_1 = x$ and $p_2 = y$ be the homogeneous polynomials of degree 1 uniquely associated with the spherical harmonics $s_1 = \cos\theta$ and $s_2 = \sin\theta$, respectively. In other words: $s_i = \mathrm{Tr}|_{\mathbb{S}^1}(p_i)$. Multiply (3.82) by $p_i\circ\mathcal{S}^m$ and integrate over Ω. Here $\mathcal{S}^m:\Omega\to\Omega$ is a diffeomorphism with the property $\mathcal{S}^m|_{\Gamma^m} = (\phi^m)^{-1}$.

$$\int_{\mathbb{S}^1} R^m f^{m+1}_t s_i + \epsilon f^{m+1}_{\theta^4 t} s_i\,d\theta = \int_\Omega u^{m+1}_t p_i\circ\mathcal{S}^m + \int_\Omega \nabla u^{m+1}\cdot\nabla\big(p_i\circ\mathcal{S}^m\big).$$

Hence

$$(1+\epsilon)\int_{\mathbb{S}^1} f^{m+1}_t s_i\,d\theta = -\int_{\mathbb{S}^1} f^m f^{m+1}_t s_i\,d\theta + \int_\Omega u^{m+1}_t p_i\circ\mathcal{S}^m + \int_\Omega \nabla u^{m+1}\cdot\nabla\big(p_i\circ\mathcal{S}^m\big). \quad (3.116)$$

Let $1\le k\le l$ and apply the differential operator $\partial_{t^{k-1}}$ to both sides of (3.116). Applying the Leibniz product rule and the Cauchy-Schwarz inequality, we arrive at:

$$\Big|\int_{\mathbb{S}^1} f^{m+1}_{t^k} s_i\,d\xi\Big| \le C\sum_{j=0}^{k-1}\big\{||u^{m+1}_{t^{j+1}}||_{L^2(\Omega^m)} + ||\nabla u^{m+1}_{t^j}||_{L^2(\Omega^m)}\big\} + C\sum_{j'=0}^{k-1}||f^m_{t^{j'}}||_{L^2}\sum_{j=1}^{k}||f^{m+1}_{t^j}||_{L^2}. \quad (3.117)$$

Observe that, due to the assumed bound on $\int_0^t D^m_{\alpha,\nu}$, we have $\sup_{0\le s\le t}||f^m||^2_{H^{2l}} \le C_A(\theta + \frac{\theta}{\alpha\nu^{l-1}})$. Hence,

$$\sum_{j'=0}^{k-1}||f^m_{t^{j'}}||_{L^2} \le ||(\mathbb{P}_0+\mathbb{P}_1)f^m||_{L^2} + \sum_{j=1}^{k-1}||(\mathbb{P}_0+\mathbb{P}_1)f^m_{t^j}||_{L^2} + (\mathcal{E}^m)^{1/2} \le CC_A^{1/2}\Big(\big(\frac{\theta}{\alpha\nu^{l-1}}\big)^{1/2} + \theta^{1/2}\Big) + C_a^{1/2}\theta^{1/2} + \theta^{1/2}$$
$$= \Big(C\big(\frac{C_A}{\alpha\nu^{l-1}}\big)^{1/2} + 1 + C_a^{1/2}\Big)\theta^{1/2} =: \mu. \quad (3.118)$$

The estimates (3.117) and (3.118) give

$$\Big|\int_{\mathbb{S}^1} f^{m+1}_{t^k} s_i\,d\xi\Big| \le C\sum_{j=0}^{k-1}\big\{||u^{m+1}_{t^{j+1}}||_{L^2(\Omega^m)} + ||\nabla u^{m+1}_{t^j}||_{L^2(\Omega^m)}\big\} + C\mu\sum_{j=1}^{k}||f^{m+1}_{t^j}||_{L^2}. \quad (3.119)$$

In the next step we exploit the stability condition $\zeta > 0$. Note that for any $k\ge1$ we have:

$$\int_{\Omega^m}(u^{m+1}_{t^k})^2 - \int_{\mathbb{S}^1}(f^{m+1}_{t^k})^2 = \int_\Omega(\mathbf{P}u^{m+1}_{t^k})^2\,dx + |\Omega|(\mathbf{P}_0 u^{m+1}_{t^k})^2 - \int_{\mathbb{S}^1}(\mathbb{P}f^{m+1}_{t^k})^2 - |\mathbb{S}^1|(\mathbb{P}_0 f^{m+1}_{t^k})^2 =$$
$$\int_\Omega(\mathbf{P}u^{m+1}_{t^k})^2\,dx - \int_{\mathbb{S}^1}(\mathbb{P}f^{m+1}_{t^k})^2 + \big(\frac{1}{|\Omega|} - \frac{1}{|\mathbb{S}^1|}\big)\big(\int_\Omega u^{m+1}_{t^k}\big)^2 + \frac{1}{|\mathbb{S}^1|}\Big(\big(\int_\Omega u^{m+1}_{t^k}\big)^2 - \big(\int_{\mathbb{S}^1} f^{m+1}_{t^k}\big)^2\Big). \quad (3.120)$$

To bound the last term on the right-most side of (3.120), we shall use the conservation law (3.97). Namely, applying the differential operator $\partial_{t^{k-1}}$ to (3.97), we have

$$\partial_{t^{k-1}}\Big(\int_\Omega u_t^{m+1}\Big)+\int_{\mathbb{S}^1} f_{t^k}^{m+1}=-\int_{\mathbb{S}^1}\partial_{t^{k-1}}(f^m f_t^{m+1}). \tag{3.121}$$

To extract the highest order contribution from the first term on LHS of (3.121) we observe the following differentiation rule for a function $\mathcal{U}$ defined on some moving domain $\Omega(t)$:

$$\partial_t\int_{\Omega(t)}\mathcal{U}=\int_{\Omega(t)}\mathcal{U}_t+\int_{\partial\Omega(t)}V\mathcal{U}, \tag{3.122}$$

where V stands for the normal velocity of the moving boundary $\partial\Omega(t)$. Using (3.122) $(k-1)$ times consecutively, it is easy to prove the formula

$$\partial_{t^{k-1}}\Big(\int_{\Omega^m} u_t^{m+1}\Big)=\int_\Omega u_{t^k}^{m+1}+\sum_{q=1}^{k-1}\partial_{t^{k-1-q}}\Big(\int_{\Gamma^m}V_{\Gamma^m}[u_{t^q}^{m+1}]_-^+\Big)=\int_\Omega u_{t^k}^{m+1}+B_k(V_{\Gamma^m},u^{m+1}), \tag{3.123}$$

where $B_k(V_{\Gamma^m},u^{m+1}):=\sum_{q=1}^{k-1}\partial_{t^{k-1-q}}\Big(\int_{\Gamma^m}V_{\Gamma^m}[u_{t^q}^{m+1}]_-^+\Big)$. Hence, from (3.121) and (3.123), we obtain

$$\int_\Omega u_{t^k}^{m+1}+\int_{\mathbb{S}^1} f_{t^k}^{m+1}=-\int_{\mathbb{S}^1}\partial_{t^{k-1}}(f^m f_t^{m+1})-B_k(V_{\Gamma^m},u^{m+1})=-B_k^m, \tag{3.124}$$

where $B_k^m:=\int_{\mathbb{S}^1}\partial_{t^{k-1}}(f^m f_t^{m+1})+B_k(V_{\Gamma^m},u^{m+1})$. Hence, using (3.124), we obtain

$$\begin{aligned}&\Big(\int_\Omega u_{t^k}^{m+1}\Big)^2-\Big(\int_{\mathbb{S}^1} f_{t^k}^{m+1}\Big)^2=\Big(\int_\Omega u_{t^k}^{m+1}-\int_{\mathbb{S}^1} f_{t^k}^{m+1}\Big)\Big(\int_\Omega u_{t^k}^{m+1}+\int_{\mathbb{S}^1} f_{t^k}^{m+1}\Big)\\&=-B_k^m\Big(2\int_\Omega u_{t^k}^{m+1}+B_k^m\Big)=-2B_k^m\int_\Omega u_{t^k}^{m+1}-(B_k^m)^2.\end{aligned} \tag{3.125}$$

By the product rule and the Cauchy-Schwarz inequality:

$$\sum_{k=1}^l\Big|\int_{\mathbb{S}^1}\partial_{t^{k-1}}(f^m f_t^{m+1})\Big|\le C\sum_{k=1}^l\sum_{j'=0}^{k-1}||f_{t^{j'}}^m||_{L^2(\mathbb{S}^1)}\sum_{j=1}^k||f_{t^j}^{m+1}||_{L^2(\mathbb{S}^1)}\le C\mu\sum_{j=1}^l||f_{t^j}^{m+1}||_{L^2}. \tag{3.126}$$

In an analogous fashion we show

$$|B_k|\le C\sum_{p=1}^{l-1}||f_{t^p}^m||_{L^2}\sum_{j=1}^{k-1}||u_{t^j}^{m+1}||_{H^1(\Omega^m)}\le C\mu\sum_{j=1}^{k-1}||u_{t^j}^{m+1}||_{H^1(\Omega^m)}. \tag{3.127}$$

Furthermore, from (3.124) we obtain for any $1\le k\le l$

$$||\mathbb{P}_0 f_{t^k}^{m+1}||_{L^2}\le C||u_{t^k}^{m+1}||_{L^2(\Omega)}+C\Big|\int_{\mathbb{S}^1}\partial_{t^{k-1}}(f^m f_t^{m+1})+B_k\Big|. \tag{3.128}$$

Now we may use (3.126), (3.127) and the definition of B_j^m to conclude

$$\sum_{k=1}^l|B_k^m|\le\sum_{k=1}^l\Big|\int_{\mathbb{S}^1}\partial_{t^{k-1}}(f^m f_t^{m+1})\Big|+\sum_{k=1}^l|B_k|\le C\mu\sum_{k=1}^l||f_{t^k}^{m+1}||_{L^2}+C\mu\sum_{k=1}^{l-1}||u_{t^k}^{m+1}||_{H^1(\Omega^m)}.$$

Using this observation together with (3.125), we conclude that

$$\begin{aligned}&\sum_{k=1}^l\Big|\Big(\int_\Omega u_{t^k}^{m+1}\Big)^2-\Big(\int_{\mathbb{S}^1} f_{t^k}^{m+1}\Big)^2\Big|\le\sum_{k=1}^l\Big\{\lambda||u_{t^k}^{m+1}||_{L^2(\Omega^m)}^2+C(1+\frac{1}{\lambda})|B_k^m|^2\Big\}\le\lambda\sum_{k=1}^l||u_{t^k}^{m+1}||_{L^2(\Omega^m)}^2\\&+C\mu^2\Big(\sum_{k=1}^l||f_{t^k}^{m+1}||_{L^2}^2+\sum_{k=1}^{l-1}||u_{t^k}^{m+1}||_{H^1(\Omega^m)}^2\Big).\end{aligned} \tag{3.129}$$

Analogously we prove ($l\to l-1$)

$$\sum_{k=1}^{l-1}\Big|\big(\int_\Omega u_{t^k}^{m+1}\big)^2-\big(\int_{\mathbb{S}^1} f_{t^k}^{m+1}\big)^2\Big|\le C(\lambda+\nu^2)\sum_{k=1}^{l-1}\Big\{||u_{t^k}^{m+1}||_{H^1(\Omega^m)}^2+||f_{t^k}^{m+1}||_{L^2}^2\Big\}. \tag{3.130}$$

Note that (3.126), (3.127) and (3.128) in particular imply

$$\sum_{j=1}^{l}||\mathbb{P}_0 f_{t^j}^{m+1}||_{L^2}^2\le C\sum_{j=1}^{l}\Big\{||u_{t^j}^{m+1}||_{L^2(\Omega^m)}^2+||\mathbb{P}_1 f_{t^j}^{m+1}||_{L^2}^2+||\mathbb{P}_{2+}f_{t^j}^{m+1}||_{L^2}^2\Big\}+C\sum_{j=1}^{l-1}||u_{t^j}^{m+1}||_{H^1(\Omega^m)}^2. \tag{3.131}$$

Having estimated the last term in (3.120), we now deal with the unpleasant negative term $-\int_{\mathbb{S}^1}(\mathbb{P}f_{t^k}^{m+1})^2$. First fundamental observation is that the Wirtinger inequality implies:

$$\int_{\mathbb{S}^1}|\omega_{\theta t^k}|^2-\int_{\mathbb{S}^1}|\mathbb{P}\omega_{t^k}|^2\ge C_W||\mathbb{P}_{2+}\omega_{\theta t^k}||_{L^2(\mathbb{S}^1)}^2.$$

Even though this would suffice to establish positivity of the energy, it features only the second and higher-order modes of ω on the right-hand side. However, the inequality (3.117) gives the obvious bound on the L^2-norm of the first modes of $f_{t^k}^{m+1}$. Namely, using the smallness of θ and ν, together with (3.131), we obtain from (3.117):

$$\sum_{k=1}^{l}||\mathbb{P}_1 f_{t^k}^{m+1}||_{L^2(\mathbb{S}^1)}^2\le C\sum_{k=1}^{l}\Big\{||u_{t^k}^{m+1}||_{L^2(\Omega^m)}^2+||\nabla u_{t^{k-1}}^{m+1}||_{L^2(\Omega^m)}^2+\mu||\mathbb{P}_{2+}f_{t^k}^{m+1}||_{L^2}^2\Big\}. \tag{3.132}$$

With θ small enough and C_a larger if necessary, from (3.131) and (3.132) we immediately deduce the second claim of (3.114). An analogous proof, using (3.130) instead of (3.129) gives the first claim of (3.114).

Proof of the second claim of (3.115). Note that the equation (3.85) can be rewritten in the form:

$$f_t^{m+1}+\epsilon f_{\theta^4 t}^{m+1}=|g^m|[u_n^{m+1}]_-^+-f_t^{m+1}f^m. \tag{3.133}$$

Let $q\le 2l-4$. Apply the differential operator ∂_{θ^q} to the equation (3.133), take the square, integrate over $\mathbb{S}^1$ and then integrate in time. We readily obtain

$$\int_0^t\Big(||f_{\theta^q t}^{m+1}||_{L^2}^2+2\epsilon||f_{\theta^{q+2}t}^{m+1}||_{L^2}^2+\epsilon^2||f_{\theta^{q+4}t}^{m+1}||_{L^2}^2\Big)\le 2\int_0^t\int_{\mathbb{S}^1}\big[\partial_{\theta^q}\big(|g^m|[u_n^{m+1}]_-^+\big)\big]^2+2\int_0^t\int_{\mathbb{S}^1}\big[\partial_{\theta^q}\big(f_t^{m+1}f^m\big)\big]^2 \tag{3.134}$$

Using the Leibniz rule, L^2-L^∞ type estimates and the trace inequality, we easily conclude that there exists a constant C_1, such that

$$\epsilon^2\int_0^t||f_{\theta^{q+4}t}^{m+1}||_{L^2}^2\le C_1\sup_{0\le s\le t}||f^m||_{H^{2l-4}}^2\int_0^t\mathfrak{D}^{m+1}. \tag{3.135}$$

On the other hand, since $\partial_t||f_{\theta^q}^{m+1}||_{L^2}\le||f_{\theta^q t}^{m+1}||_{L^2}$, using the smallness assumptions in Lemma 3.2, we easily deduce

$$\epsilon^2||f_{\theta^{q+4}}^{m+1}||_{L^2}^2\le\epsilon^2||f_{\theta^{q+4}}^{m+1}(0)||_{L^2}^2+C_1C_A(\theta_1+\theta)t\int_0^t\mathfrak{D}^{m+1}.$$

Thus, using the above for any $0\le q\le 2l-4$, assuming that $||f_0||_{H^{2l}}^2\le K\theta_1$ for some constant $K>0$ and choosing θ,θ_1 such that $(\theta+\theta_1)C_1,K\theta_1\le 1$ suitably small, we obtain

$$||f^{m+1}||_{H^{2l}}^2\le C_A\big(\theta_1+\frac{t}{\epsilon^2}\int_0^t\mathfrak{D}^{m+1}\big) \tag{3.136}$$

and this proves the second claim of (3.115). If we apply the operator ∂_{θ^q} with $q\le 2l-2$ to the equation (3.133), in the same way as above we get (with no ϵ!)

$$||f^{m+1}||_{H^{2l-2}}^2\le C_A\big(\theta_1+t\int_0^t\mathfrak{D}^{m+1}\big) \tag{3.137}$$

Note that the same proof also gives the bound

$$||f^{m+1}||_{H^{2l-2}}^2\le C_A\big(\theta_1+t^2\sup_{0\le s\le t}\mathfrak{E}^{m+1}(s)\big). \tag{3.138}$$

3.2.2 Positive definiteness

Let $\eta>0$ be a small number to be specified later. Using the definition (3.99) of $\mathcal{E}^m$ and combining (3.120), (3.129) and (3.132), we have

$$\begin{aligned}(1+\eta)\mathcal{D}^{m+1}\geq&\sum_{k=1}^{l}\Big\{(1+\eta)\Big[\int_\Omega(\mathbf{P}u_{t^k}^{m+1})^2+\int_\Omega|\nabla u_{t^{k-1}}^{m+1}|^2+\zeta\big(\int_\Omega u_{t^k}^{m+1}\big)^2+\epsilon C_W||\mathbb{P}_{2+}f_{\theta^3t^k}^{m+1}||_{L^2}^2\\&-C\lambda||u_{t^k}^{m+1}||_{L^2(\Omega^m)}^2-C\mu^2\Big(||u_{t^k}^{m+1}||_{L^2(\Omega)}^2+||\nabla u_{t^{k-1}}^{m+1}||_{L^2}^2+||\mathbb{P}_1f_{t^k}^{m+1}||_{L^2}^2+||\mathbb{P}_{2+}f_{t^k}^{m+1}||_{L^2}^2\Big)\Big]\\&+C_W||\mathbb{P}_{2+}f_{\theta t^k}^{m+1}||_{L^2}^2+\eta||f_{\theta t^k}^{m+1}||_{L^2}^2-\eta C\sum_{j=1}^{k}\Big\{||u_{t^j}^{m+1}||_{L^2(\Omega^m)}^2+||\nabla u_{t^{j-1}}^{m+1}||_{L^2(\Omega^m)}^2+\nu||\mathbb{P}_{2+}f_{t^j}^{m+1}||_{L^2}^2\Big\}.\end{aligned}$$

After regrouping, we obtain

$$\begin{aligned}&(1+\eta)\mathcal{D}^{m+1}+C\sum_{k=1}^{l}\Big\{\mu^2\Big(||u_{t^k}^{m+1}||_{L^2(\Omega^m)}^2+||\nabla u_{t^{k-1}}^{m+1}||_{L^2(\Omega^m)}^2+||\mathbb{P}_1f_{t^k}^{m+1}||_{L^2}^2+||\mathbb{P}_{2+}f_{t^k}^{m+1}||_{L^2}^2\Big)\\&+C\lambda\sum_{k=1}^{l}||u_{t^k}^{m+1}||_{L^2(\Omega^m)}^2+\eta\sum_{j=1}^{k}\Big\{||u_{t^j}^{m+1}||_{L^2(\Omega^m)}^2+||\nabla u_{t^{j-1}}^{m+1}||_{L^2(\Omega^m)}^2+\mu||\mathbb{P}_{2+}f_{t^j}^{m+1}||_{L^2}^2\Big\}\Big\}\\&\geq C'\sum_{k=1}^{l}\Big\{\epsilon C_W||\mathbb{P}_{2+}f_{\theta^3t^k}^{m+1}||_{L^2}^2+||u_{t^k}^{m+1}||_{L^2(\Omega^m)}^2+||\nabla u_{t^{k-1}}^{m+1}||_{L^2(\Omega^m)}^2+\eta||f_{\theta t^k}^{m+1}||_{L^2}^2+C_W||\mathbb{P}_{2+}f_{\theta t^k}^{m+1}||_{L^2}^2\Big\}.\end{aligned}$$

First choosing η and then λ, θ, μ and ϵ small, we obtain

$$\mathcal{D}^{m+1}\geq C''\sum_{k=1}^{l}\Big\{||u_{t^k}^{m+1}||_{L^2(\Omega)}^2+||\nabla u_{t^{k-1}}^{m+1}||_{L^2(\Omega)}^2+||f_{\theta t^k}^{m+1}||_{L^2}^2+\epsilon||\mathbb{P}_{2+}f_{\theta^3t^k}^{m+1}||_{L^2}^2\Big\}\tag{3.139}$$

Let us denote

$$\mathcal{D}_{\|}(u,f):=\sum_{k=1}^{l}\Big\{||u_{t^k}||_{L^2(\Omega^m)}^2+||\nabla u_{t^{k-1}}||_{L^2(\Omega^m)}^2+||f_{\theta t^k}||_{L^2}^2+\epsilon||f_{\theta^3t^k}||_{L^2}^2\Big\}$$

Using (3.139) together with (3.131) and the definition (3.99) of $\mathcal{D}^j$, we conclude that there exists a constant $C>0$ such that

$$\frac{1}{C}\mathcal{D}_{\|}(u^{m+1},f^{m+1})\leq\mathcal{D}^{m+1}\leq C\mathcal{D}_{\|}(u^{m+1},f^{m+1}).\tag{3.140}$$

Similarly we introduce

$$\mathcal{E}_{\|}(u,f):=\sum_{k=0}^{l-1}\big\{||u_{t^k}||_{L^2(\Omega^m)}^2+||\nabla u_{t^k}||_{L^2(\Omega^m)}^2\big\}+||\mathbb{P}_0f||_{L^2}^2+||\mathbb{P}_2f||_{L^2}^2+\sum_{k=1}^{l-1}\big\{||f_{\theta t^k}||_{L^2}^2+\epsilon||f_{\theta^3t^k}||_{L^2}^2\big\}\tag{3.141}$$

A proof analogous to the proof of (3.139), which uses (3.130) instead of (3.129) leads to

$$\mathcal{E}^{m+1}\geq\mathcal{E}_{(0)}^{m+1}+C_3\sum_{k=1}^{l-1}\Big\{||u_{t^k}^{m+1}||_{L^2(\Omega^m)}^2+||\nabla u_{t^k}^{m+1}||_{L^2(\Omega^m)}^2+||f_{\theta t^k}^{m+1}||_{L^2}^2+\epsilon||\mathbb{P}_{2+}f_{\theta^3t^k}^{m+1}||_{L^2}^2.\Big\}\tag{3.142}$$

Like above we conclude that there exists a constant $C>0$ such that

$$\frac{1}{C}\mathcal{E}_{\|}(u^{m+1},f^{m+1})\leq\mathcal{E}^{m+1}\leq C\mathcal{E}_{\|}(u^{m+1},f^{m+1}).\tag{3.143}$$

3.2.3 Temporal energy estimates

The last and the hardest estimate in the statement of Lemma 3.2 is the first claim of (3.115). We first estimate RHS of the zero-th order energy identity (3.112). We then collect a number of useful estimates in Lemma 3.3 and then use it to estimate RHS of the higher-order energy identity (3.101).

We start off by estimating RHS of the identity (3.112). Note that $A(u^{m+1},f^m,f^{m+1})$ is given by the formula (3.111). The first term is a cross-term and is easily estimated as follows:

$$\begin{aligned}&\Big|\int_0^t\int_{\mathbb{S}^1}(f^m-f^{m+1}+f^m_{\theta\theta}-f^{m+1}_{\theta\theta})f^{m+1}_t\Big|\le\lambda\int_0^t||f^{m+1}_t||^2_{H^1}+\frac{C_0}{\lambda}\int_0^t\big(||(I-\mathbb{P}_1)f^m||^2_{H^1}+||(I-\mathbb{P}_1)f^{m+1}||^2_{H^1}\big)\\&\le\lambda\int_0^t\mathcal{D}^{m+1}+\frac{C_0t}{\lambda}C_A(2\theta_1+\int_0^t\mathfrak{D}^m+\int_0^t\mathfrak{D}^{m+1}),\end{aligned}\tag{3.144}$$

where we used the second claim of (3.115) and the assumption that $t<1$. As to the second term on RHS of (3.111), note that from the assumptions of Lemma 3.2 we have

$$\Big|\int_0^t\int_{\mathbb{S}^1}f^{m+1}_tN_1(f^m)\Big|\le\lambda\int_0^t\int_{\mathbb{S}^1}(f^{m+1}_t)^2+\frac{C_0t}{\lambda}\sup_{0\le s\le t}||N_1(f^m)||^2_{L^2}\le\lambda\int_0^t\mathcal{D}^{m+1}+\frac{C_0C_At}{\lambda}\big(\theta_1+\int_0^t\mathfrak{D}^m\big)^2\tag{3.145}$$

In order to deal with the third term on RHS of (3.112), we first use the conservation law (3.98) to expand that expression:

$$\begin{aligned}&\partial_t\Big\{\big(\int_{\Omega^m}u^{m+1}\big)^2-\big(\int_{\mathbb{S}^1}f^{m+1}\big)^2\Big\}=\big(\int_{\Omega^m}u^{m+1}_t+\int_{\mathbb{S}^1}f^{m+1}_t\big)\big(\int_{\Omega^m}u^{m+1}-\int_{\mathbb{S}^1}f^{m+1}\big)\\&+\big(\int_{\Omega^m}u^{m+1}+\int_{\mathbb{S}^1}f^{m+1}\big)\big(\int_{\Omega^m}u^{m+1}_t-\int_{\mathbb{S}^1}f^{m+1}_t\big)=-\int_{\mathbb{S}^1}f^{m+1}_tf^m\times\big(\int_{\Omega^m}u^{m+1}-\int_{\mathbb{S}^1}f^{m+1}\big)\\&-\Big(\int_{\mathbb{S}^1}\frac{(f^{m+1})^2}{2}+\int_0^t\int_{\mathbb{S}^1}f^{m+1}_t(f^m-f^{m+1})\Big)\Big(\int_{\Omega^m}u^{m+1}_t-\int_{\mathbb{S}^1}f^{m+1}_t\Big).\end{aligned}\tag{3.146}$$

Using the inequality between the geometric and arithmetic mean and (3.146), we easily deduce

$$\begin{aligned}&\Big|\int_0^t\partial_t\Big\{\big(\int_{\Omega^m}u^{m+1}\big)^2-\big(\int_{\mathbb{S}^1}f^{m+1}\big)^2\Big\}\Big|\le\lambda\int_0^t\int_{\mathbb{S}^1}(f^{m+1}_t)^2+\lambda\int_0^t\int_{\Omega^m}(u^{m+1}_t)^2\\&+\frac{C_0t}{\lambda}\sup_{0\le s\le t}\big(||u^{m+1}||^2_{L^2(\Omega)}+||f^m||^2_{L^2}+||f^{m+1}||^2_{L^2}+||f^{m+1}||^4_{L^2}\big)\\&\le\lambda\int_0^t\mathcal{D}^{m+1}+\frac{C_0t}{\lambda}\sup_{0\le s\le t}\mathfrak{E}^{m+1}(s)+\frac{C_0t}{\lambda}C_A\big(\theta_1+\int_0^t\mathfrak{D}^m+\int_0^t\mathfrak{D}^{m+1}+\sup_{0\le s\le t}(\mathfrak{E}^{m+1})^2(s)\big),\end{aligned}\tag{3.147}$$

where we used both (3.136) and (3.138), combined with the assumption $t<1$. The last term on RHS of (3.112) is $\int_0^t\int_{\mathbb{S}^1}A^m$, where A^m is given by (3.106). The first two expressions in the definition of A^m are the cross-terms. We first use the integration by parts and then reason similarly to (3.144):

$$\begin{aligned}&\epsilon\Big|\int_0^t\int_{\mathbb{S}^1}-f^{m+1}_{\theta^2t}(f^{m+1}_{\theta^2}-f^m_{\theta^2})+f^{m+1}_{\theta^3t}(f^{m+1}_{\theta^3}-f^m_{\theta^3})\Big|=\epsilon\Big|\int_0^t\int_{\mathbb{S}^1}f^{m+1}_{\theta^3t}\big(f^{m+1}_\theta+f^{m+1}_{\theta^3}-(f^m_\theta+f^m_{\theta^3})\big)\Big|\\&\le\lambda\big(\epsilon\int_0^t||f^{m+1}_{\theta^3t}||^2_{L^2}+\frac{C_0}{\lambda}\big(\epsilon\int_0^t||(I-\mathbb{P}_1)f^m||^2_{H^3}+\epsilon\int_0^t||(I-\mathbb{P}_1)f^{m+1}||^2_{H^3}\big)\le\lambda\Big(\int_0^t\hat{\mathcal{D}}^{m+1}+\epsilon\int_0^t\mathcal{D}^{m+1}\Big)\\&+\frac{C_0C_At}{\lambda}\big(\theta_1+\int_0^t\mathfrak{D}^m+\int_0^t\mathfrak{D}^{m+1}\big).\end{aligned}\tag{3.148}$$

As to the third term on RHS of (3.106), we start by integrating by parts:

$$
\begin{aligned}
&\epsilon\Big|\int_0^t\int_{\mathbb{S}^1} f^m_{\theta^2}f^{m+1}_{\theta^4 t}\big(\frac{1}{R^m|g^m|}-1\big)\Big| \le \epsilon\Big|\int_0^t\int_{\mathbb{S}^1} f^m_{\theta^3}f^{m+1}_{\theta^3 t}\big(\frac{1}{R^m|g^m|}-1\big)\Big| + \Big|\int_0^t\int_{\mathbb{S}^1} f^m_{\theta^2}f^{m+1}_{\theta^3 t}\big(\frac{1}{R^m|g^m|}\big)_\theta\Big| \\
&\le \lambda\epsilon\int_0^t ||f^{m+1}_{\theta^3 t}||^2_{L^2} + \frac{C_0 t}{\lambda}\epsilon\sup_{0\le s\le t}\big(||\frac{1}{R^m|g^m|}-1||^2_{L^\infty}||f^m_{\theta^3}||^2_{L^2}\big) + \lambda\epsilon\int_0^t ||f^{m+1}_{\theta^3 t}||^2_{L^2} \\
&+\frac{C_0 t}{\lambda}\epsilon\sup_{0\le s\le t}\big(||\big(\frac{1}{R^m|g^m|}\big)_\theta||^2_{L^\infty}||f^m_{\theta^2}||^2_{L^2}\big) \le \lambda\big(\int_0^t \hat{\mathcal{D}}^{m+1} + \epsilon\int_0^t \mathcal{D}^{m+1}\big) + \frac{C_0 t}{\lambda}\sup_{0\le s\le t}\big(||\mathbb{P}f^m||^2_{H^2}\epsilon||\mathbb{P}f^m_{\theta^2}||^2_{H^1}\big) \\
&\le \lambda\big(\int_0^t \hat{\mathcal{D}}^{m+1} + \epsilon\int_0^t \mathcal{D}^{m+1}\big) + \frac{C_0 t}{\lambda}\sup_{0\le s\le t}||f^m||_{H^2}C_A\big(\theta_1 + \int_0^t \mathfrak{D}^m\big),
\end{aligned}
\tag{3.149}
$$

where we recall the definitions (2.46) and (1.33) of $\hat{\mathcal{D}}^\epsilon$ and $\mathcal{D}$ respectively. In the last estimate we used the smallness assumption on $||f^m||_{H^{2l}}$ of Lemma 3.2. Finally, the last term on RHS of (3.106) can be estimated analogously to (3.149) and we obtain:

$$
\epsilon\Big|\int_0^t\int_{\mathbb{S}^1} N^*(f^m)f^{m+1}_{\theta^4 t}\Big| \le \lambda\int_0^t \mathcal{D}^{m+1} + \frac{C_0 t}{\lambda}\sup_{0\le s\le t}||f^m||_{H^2}C_A\big(\theta_1 + \int_0^t \mathfrak{D}^m\big). \tag{3.150}
$$

The following lemma states a number of inequalities that will be used in estimating the remaining error terms.

Lemma 3.3 *The following inequalities hold:*

(a) For any $0\le k\le l-1$ the following inequalities hold:

$$
\begin{aligned}
&\int_0^t ||[\gamma^m,\partial_{t^{k+1}}]u^{m+1}||^2_{L^2(\Gamma^m)} \le C\sup_{0\le s\le t}\mathfrak{E}^m(s)\int_0^t\mathfrak{D}^{m+1} + \int_0^t\mathfrak{D}^m\sup_{0\le s\le t}\mathfrak{E}^{m+1}(s) \\
&\int_0^t ||[\gamma^m,\partial_{t^k}]u^{m+1}||^2_{H^1(\Gamma^m)},\ \int_0^t ||[\gamma^m,\partial_{t^k}]u^{m+1}_n||^2_{L^2(\Gamma^m)} \le C\sup_{0\le s\le t}\mathfrak{E}^m(s)\int_0^t\mathfrak{D}^{m+1}.
\end{aligned}
$$

(b) For any $0\le k\le l-1$, the following inequality holds (∂_t^* is defined in (2.58)):

$$
\int_0^t ||\partial^*_{t^k}V^{m+1}||^2_{L^2(\Gamma^m)} \le C\sup_{0\le s\le t}(1+\mathfrak{E}^m(s))\int_0^t\mathcal{D}^{m+1}.
$$

(c) Let $\xi\in C^\infty(J,\mathbb{R})$ and $J\subseteq\mathbb{R}$ an interval such that every derivative of ξ is uniformly bounded on J. Let $0<r<l$. Then there exists a positive constant C such that

$$
||\partial_{t^r}\big[\xi((R^m)^2+(f^m_\theta)^2)\big]||_{H^1} \le C\big(\sqrt{\mathcal{E}^m}+\sqrt{\mathfrak{E}^m}\big) \tag{3.151}
$$

and

$$
||\partial_{t^r}\big[\xi((R^m)^2+(f^m_\theta)^2)\big]||_{H^3} \le \frac{C}{\sqrt{\epsilon}}\big(\sqrt{\mathcal{E}^m}+\sqrt{\mathfrak{E}^m}\big). \tag{3.152}
$$

Proof. Recall the formula (2.61):

$$
[\gamma^m,\partial_{t^l}]u^{m+1} = \sum_{p=0}^{l-1}\sum_{q+r=l-p} C^{p,q,r}\partial^*_{t^{l-1-p}}\big(V^qV^r_\parallel\big)\partial_{s^r}\partial_{n^q}u^{m+1}_{t^p}.
$$

Note that for any $1\le p\le l-1$ $||\partial_{t^{l-1-p}}(V^qV^r_\parallel\circ\phi^m)||_{L^2}\le C\sqrt{\mathfrak{E}^m}$ and for any $q,r\in\mathbb{N}_0$ such that $q+r=l-p$, we have

$$
||\partial_{s^r}\partial_{n^q}u^{m+1}_{t^p}||_{L^\infty(\Gamma^m)} \le C||\nabla u^{m+1}_{t^p}||_{H^{l-p}(\Gamma^m)} \le C\sqrt{\mathfrak{D}^{m+1}},
$$

where we use the trace inequality and the fact that $l-p+1\le 2(l-1-p)+2$ (observe (1.37)). If $p=0$, then $||\partial_{t^{l-1}}(V^qV^r_\parallel\circ\phi^m)||_{L^2}\le C\mathfrak{E}^m\sqrt{\mathfrak{D}^m}$ and from the trace inequality, (1.35) and the fact that $q+r=l$:

$$
||\partial_{s^r}\partial_{n^q}u^{m+1}||_{L^\infty(\Gamma^m)} \le C||u^{m+1}||_{H^{l+1}(\Gamma^m)} \le C\sqrt{\mathfrak{E}^{m+1}},
$$

where we note that $||u^{m+1}||_{H^{l+1}(\Gamma^m)} \leq C\sqrt{\mathfrak{E}^{m+1}}$ if $l \geq 2$ (observe (1.35)). Combining the above inequalities, we deduce

$$\int_0^t ||[\gamma^m, \partial_{t^{k+1}}]u^{m+1}||^2_{L^2(\Gamma^m)} \leq C \sup_{0\leq s\leq t} \mathfrak{E}^m(s) \int_0^t \mathfrak{D}^{m+1} + \int_0^t \mathfrak{D}^m \sup_{0\leq s\leq t} \mathfrak{E}^{m+1}(s)$$

and this proves the first claim of part (a) of Lemma 3.3. The other two claims are proven similarly. As to the part (b), fix $0 \leq k \leq l-1$. Then

$$\begin{aligned}
&\int_0^t ||\partial_{t^k}\big(V^{m+1}\circ\phi^m\big)||^2_{L^2} \leq C\sum_{q=0}^k \int_0^t ||f^{m+1}_{t^{q+1}}\big(\frac{R^m}{|g^m|}\big)_{t^{k-q}}||^2_{L^2} \\
&\leq C\sum_{q=0}^k \sup_{0\leq s\leq t} ||\big(\frac{R^m}{|g^m|}\big)_{t^{k-q}}||^2_{L^\infty} \int_0^t ||f^{m+1}_{t^{q+1}}||^2_{L^2} \leq C(1+\sup_{0\leq s\leq t}\mathfrak{E}^m(s)) \int_0^t \mathcal{D}^{m+1}
\end{aligned}$$

and this finishes the proof of part (b). Note that we used the Leibniz rule in the first inequality, and then standard L^∞-L^2 type inequality, combined with the Sobolev inequality. Part (c) follows using similar techniques and the definition (1.34) of $\mathfrak{E}$ and $\mathfrak{D}$. This proof is in fact analogous to the proof of part (a) of Lemma 3.3 in [21]. $\square$

We proceed now to estimate RHS of the energy identity (3.101).

Estimating $\int_0^t \int_{\Gamma^m} O^m_k$. Recall that O^m_k is defined by (3.102) and (2.73):

$$O^m_k = -[\gamma^m, \partial_{t^k}]u^{m+1}[\partial_n u^{m+1}_{t^k}]^+_- - \partial^*_{t^k} u^{m+1}[\gamma^m, \partial_{t^k}][u^{m+1}_n]^+_- + V_{\Gamma^m}(u^{m+1}_{t^k})^2. \tag{3.153}$$

In order to estimate the first term on RHS of (3.153), we apply the relation (2.61) to write

$$[\partial_n u^{m+1}_{t^k}]^+_- = \partial^*_{t^k}[u^{m+1}_n]^+_- - [\gamma^m, \partial_{t^k}][u^{m+1}_n]^+_-. \tag{3.154}$$

We next note that from (2.49)

$$\begin{aligned}
&\Big|\int_0^t \int_{\Gamma^m} [\gamma^m, \partial_{t^k}]u^{m+1}[\gamma^m, \partial_{t^k}][u^{m+1}_n]^+_-\Big| \leq \int_0^t ||[\gamma^m, \partial_{t^k}]u^{m+1}||_{L^2(\Gamma^m)} ||[\gamma^m, \partial_{t^k}][u^{m+1}_n]^+_-||_{L^2(\Gamma^m)} \\
&\leq \int_0^t ||[\gamma^m, \partial_{t^k}]u^{m+1}||^2_{L^2(\Gamma^m)} + \int_0^t ||[\gamma^m, \partial_{t^k}][u^{m+1}_n]^+_-||^2_{L^2(\Gamma^m)} \leq C\sqrt{\mathfrak{E}^m}\int_0^t \mathfrak{D}^{m+1},
\end{aligned} \tag{3.155}$$

where we used part (a) of Lemma 3.3 in the last line. Furthermore, using (3.84) yields:

$$\begin{aligned}
&\Big|\int_0^t \int_{\Gamma^m} [\gamma^m, \partial_{t^k}]u^{m+1}\partial^*_{t^k}[u^{m+1}_n]^+_-\Big| = \Big|\int_0^t \int_{\Gamma^m} [\gamma^m, \partial_{t^k}]u^{m+1}\partial^*_{t^k}\big(V^{m+1} + \epsilon\Lambda^{m+1}\big)\Big| \\
&\leq \frac{C}{\lambda}\int_0^t ||[\gamma^m, \partial_{t^k}]u^{m+1}||^2_{L^2(\Gamma^m)} + \lambda\int_0^t ||\partial^*_{t^k}V^{m+1}||^2_{L^2(\Gamma^m)} + \epsilon\Big|\int_0^t \int_{\mathbb{S}^1} [\gamma^m, \partial_{t^k}]u^{m+1}\circ\phi^m \partial_{t^k}\big(\frac{f^{m+1}_{\theta^4 t}}{|g^m|}\big)|g^m|\Big|,
\end{aligned} \tag{3.156}$$

where Λ^{m+1} is defined in the line after (3.84). Note that by parts (a) and (b) of Lemma 3.3,

$$\frac{C}{\lambda}\int_0^t ||[\gamma^m, \partial_{t^k}]u^{m+1}||^2_{L^2(\Gamma^m)} + \lambda\int_0^t ||\partial^*_{t^k}V^{m+1}||^2_{L^2(\Gamma^m)} \leq C\sqrt{\mathfrak{E}^m}\int_0^t \mathfrak{D}^{m+1}. \tag{3.157}$$

Furthermore, recall the formula $\partial_{t^k}\big(\frac{f^{m+1}_{\theta^4 t}}{|g^m|}\big) = \frac{f^{m+1}_{\theta^4 t^{k+1}}}{|g^m|} + h^m_{k,2}$ ($h^m_{k,2}$ given by (3.96)). Hence

$$\begin{aligned}
&\epsilon\Big|\int_0^t \int_{\mathbb{S}^1} [\gamma^m, \partial_{t^k}]u^{m+1}\circ\phi^m \partial_{t^k}\big(\frac{f^{m+1}_{\theta^4 t}}{|g^m|}\big)|g^m|\Big| = \epsilon\Big|\int_0^t \int_{\mathbb{S}^1} [\gamma^m, \partial_{t^k}]u^{m+1}\circ\phi^m\big(f^{m+1}_{\theta^4 t^{k+1}} + h^m_{k,2}\big)|g^m| \\
&\leq \frac{C}{\lambda}\int_0^t ||[\gamma^m, \partial_{t^k}]u^{m+1}||^2_{H^1(\Gamma^m)} + \lambda\epsilon^2\int_0^t \int_{\mathbb{S}^1} ||f^{m+1}_{\theta^3 t^{k+1}}||^2_{L^2} + \epsilon\int_0^t \int_{\mathbb{S}^1} [\gamma^m, \partial_{t^k}]u^{m+1}\circ\phi^m h^m_{k,2}|g^m|
\end{aligned} \tag{3.158}$$

Note that by (3.96) and integration by parts the last term above is bounded as follows:

$$
\begin{aligned}
&\epsilon\Big|\int_0^t\int_{\mathbb{S}^1}[\gamma^m,\partial_{t^k}]u^{m+1}\circ\phi^m h^m_{k,2}|g^m|\Big| = \epsilon\Big|\sum_{q=0}^{k-1}\int_0^t\int_{\mathbb{S}^1}\partial^4_{q+1}f^{m+1}\big(\frac{1}{|g^m|}\big)_{t^{k-q}}[\gamma^m,\partial_{t^k}]u^{m+1}\circ\phi^m h^m_{k,2}|g^m|\\
&=\epsilon\Big|\sum_{q=0}^{k-1}\int_0^t\int_{\mathbb{S}^1}\partial^3_{q+1}f^{m+1}\big(\frac{1}{|g^m|}\big)_{\theta t^{k-q}}[\gamma^m,\partial_{t^k}]u^{m+1}\circ\phi^m h^m_{k,2}|g^m|\\
&+\sum_{q=0}^{k-1}\int_0^t\int_{\mathbb{S}^1}\partial^3_{q+1}f^{m+1}\big(\frac{1}{|g^m|}\big)_{t^{k-q}}\partial_\theta\big([\gamma^m,\partial_{t^k}]u^{m+1}\circ\phi^m h^m_{k,2}|g^m|\big)\Big| \le \sum_{q=0}^{k-1}\int_0^t\epsilon||\partial^3_{q+1}f^{m+1}\big(\frac{1}{|g^m|}\big)_{\theta t^{k-q}}||_{L^2}\times\\
&\times||[\gamma^m,\partial_{t^k}]u^{m+1}\circ\phi^m h^m_{k,2}|g^m|||_{L^2}+\sum_{q=0}^{k-1}\int_0^t\epsilon||\partial^3_{q+1}f^{m+1}\big(\frac{1}{|g^m|}\big)_{t^{k-q}}||_{L^2}||[\gamma^m,\partial_{t^k}]u^{m+1}\circ\phi^m h^m_{k,2}|g^m|||_{L^2}
\end{aligned}
\tag{3.159}
$$

Note that from (2.49)

$$
\epsilon||\partial^3_{q+1}f^{m+1}\big(\frac{1}{|g^m|}\big)_{\theta t^{k-q}}||_{L^2} \le \sqrt{\epsilon}||\big(\frac{1}{|g^m|}\big)_{\theta t^{k-q}}||_{L^\infty}\sqrt{\epsilon}||\partial^3_{q+1}f^{m+1}||_{L^2} \le C\sqrt{\hat{\mathfrak{E}}^m}\sqrt{\hat{\mathcal{D}}^{m+1}}. \tag{3.160}
$$

Similarly $\epsilon||\partial^3_{q+1}f^{m+1}\big(\frac{1}{|g^m|}\big)_{t^{k-q}}||_{L^2} \le C\sqrt{\hat{\mathfrak{E}}^m}\sqrt{\hat{\mathcal{D}}^{m+1}}$. From (3.159), (3.160) and the estimate after (3.160), together with part (a) of Lemma 3.3, we deduce

$$
\begin{aligned}
&\epsilon\Big|\int_0^t\int_{\mathbb{S}^1}[\gamma^m,\partial_{t^k}]u^{m+1}\circ\phi^m h^m_{k,2}|g^m|\Big| \le C\sup_{0\le s\le t}\mathfrak{E}^m\int_0^t\mathcal{D}^{m+1}+\int_0^t||[\gamma^m,\partial_{t^k}]u^{m+1}||^2_{H^1(\Gamma^m)}\\
&\le C\sup_{0\le s\le t}\hat{\mathfrak{E}}^m\int_0^t\hat{\mathcal{D}}^{m+1}+C\sup_{0\le s\le t}\mathfrak{E}^m\int_0^t\mathcal{D}^{m+1}.
\end{aligned}
\tag{3.161}
$$

From (3.156) - (3.158), (3.161) and part (b) of Lemma 3.3, we finally conclude

$$
\begin{aligned}
&\Big|\int_0^t\int_{\Gamma^m}[\gamma^m,\partial_{t^k}]u^{m+1}\partial^*_{t^k}[u_n^{m+1}]^+_-\Big| \le C\sqrt{\mathfrak{E}^m}\int_0^t\mathfrak{D}^{m+1}+C\epsilon\lambda\int_0^t\hat{\mathcal{D}}^{m+1}+C\epsilon^2\lambda\int_0^t\mathcal{D}^{m+1}\\
&+C\sup_{0\le s\le t}\hat{\mathfrak{E}}^m\int_0^t\hat{\mathcal{D}}^{m+1}+C\sup_{0\le s\le t}\mathfrak{E}^m\int_0^t\mathcal{D}^{m+1}.
\end{aligned}
\tag{3.162}
$$

Estimating $\int_0^t\int_{\Gamma^m}P^m_k$. Recall that P^m_k is defined by (3.102) and (2.74):

$$
P^m_k = \big(\partial_t([\gamma^m,\partial_{t^k}]u^{m+1,\pm})-[\gamma^m,\partial_t]u^{m+1,\pm}_{t^k}\big)\partial_n u^{m+1,\pm}_{t^k}+\partial^*_{t^{k+1}}u^{m+1}[\gamma^m,\partial_{t^k}][u_n^{m+1}]^+_-+V_{\Gamma^m}|\nabla u^{m+1,\pm}|^2.
$$

It is, however, easy to see that the expression in the first parenthesis above simplifies, i.e.,

$$
\partial_t([\gamma^m,\partial_{t^k}]u^{m+1})-[\gamma^m,\partial_t]u^{m+1}_{t^k} = [\gamma^m,\partial_{t^{k+1}}]u^{m+1}.
$$

Using this observation together with (2.60) applied to $[\partial_n u^{m+1}_{t^{k-1}}]$, we may rewrite $P(\mathcal{U})$:

$$
P^m_k = [\gamma^m,\partial_{t^{k+1}}]u^{m+1}\partial_n u^{m+1,\pm}_{t^k}+\partial^*_{t^{k+1}}u^{m+1}[\gamma^m,\partial_{t^k}][u_n^{m+1}]^+_-+V_{\Gamma^m}|\nabla u^{m+1,\pm}|^2. \tag{3.163}
$$

We start by estimating the first contribution on RHS of (3.163). Note that

$$
\begin{aligned}
&\Big|\int_0^t\int_{\Gamma^m}[\gamma^m,\partial_{t^{k+1}}]u^{m+1}\partial_n u^{m+1,\pm}_{t^k}\Big| \le \int_0^t\frac{C}{\lambda}||[\gamma^m,\partial_{t^{k+1}}]u^{m+1}||^2_{L^2(\Gamma^m)}+\lambda\int_0^t||\partial_n u^{m+1,\pm}_{t^k}||^2_{L^2(\Gamma^m)}\\
&\le\frac{C}{\lambda}\sup_{0\le s\le t}\mathfrak{E}^m\int_0^t\mathfrak{D}^{m+1}+\frac{C}{\lambda}\sup_{0\le s\le t}\mathfrak{E}^{m+1}\int_0^t\mathcal{D}^m+C\lambda\int_0^t\mathfrak{D}^{m+1},
\end{aligned}
\tag{3.164}
$$

where we used part (a) of Lemma 3.3 and the trace inequality in the second inequality. As to the second term on RHS of (3.163), first recall that $\partial^*_{t^{k+1}}u^{m+1}\circ\phi^m = -f^m_{t^{k+1}}-\frac{f^m_{\theta^2 t^{k+1}}}{R^m|g^m|}+G^m_{k+1}$ (G^m_{k+1} defined by (3.95)).

First note that

$$\begin{aligned}&\Big|\int_0^t\int_{\mathbb{S}^1} f^m_{t^{k+1}}[\gamma^m,\partial_{t^k}][u_n^{m+1}]^+_-\circ\phi^m|g^m|\Big|\le\lambda\int_0^t||f^m_{t^{k+1}}||^2_{L^2}+\frac{C}{\lambda}\int_0^t||[\gamma^m,\partial_{t^k}][u_n^{m+1}]^+_-||^2_{L^2(\Gamma^m)}\\&\le\lambda\int_0^t\mathcal{D}^m+\frac{C}{\lambda}\sqrt{\mathfrak{E}^m}\int_0^t\mathfrak{D}^{m+1},\end{aligned}\tag{3.165}$$

where we used part (a) of Lemma 3.3 in the last estimate. Integrating by parts and reasoning in the same way as in (3.165), we obtain

$$\Big|\int_0^t\int_{\mathbb{S}^1}\frac{f^m_{\theta^2t^{k+1}}}{R^m|g^m|}[\gamma^m,\partial_{t^k}][u_n^{m+1}]^+_-\circ\phi^m|g^m|\Big|\le\lambda\int_0^t\mathcal{D}^m+\frac{C}{\lambda}\sqrt{\mathfrak{E}^m}\int_0^t\mathfrak{D}^{m+1}.\tag{3.166}$$

Note that by (3.95)

$$\begin{aligned}&\Big|\int_0^t\int_{\mathbb{S}^1}G^m_{k+1}[\gamma^m,\partial_{t^k}][u_n^{m+1}]^+_-\circ\phi^m|g^m|\Big|\le\Big|\int_0^t\int_{\mathbb{S}^1}N^*(f^m)_{t^{k+1}}[\gamma^m,\partial_{t^k}][u_n^{m+1}]^+_-\circ\phi^m|g^m|\Big|\\&+\sum_{q=0}^k\Big|\int_0^t\int_{\mathbb{S}^1}\partial^2_qf^m\partial_{k+1-q}(\frac{1}{R^m|g^m|})[\gamma^m,\partial_{t^k}][u_n^{m+1}]^+_-\circ\phi^m|g^m|\Big|\end{aligned}\tag{3.167}$$

The first term on RHS of (3.167) is easily estimated by the Cauchy-Schwarz inequality and part (a) of Lemma (3.3)(recall (1.34)) ($N^*(f)$ defined in the line after (3.96)):

$$\begin{aligned}&\Big|\int_0^t\int_{\mathbb{S}^1}N^*(f^m)_{t^{k+1}}[\gamma^m,\partial_{t^k}][u_n^{m+1}]^+_-\circ\phi^m|g^m|\Big|\le\int_0^t||N^*(f^m)_{t^{k+1}}||^2_{L^2}+\int_0^t||[\gamma^m,\partial_{t^k}][u_n^{m+1}]^+_-||^2_{L^2}\\&\le C\sup_{0\le s\le t}\big(||f^m(s)||^2_{H^3}+\mathfrak{E}^m(s)\big)\int_0^t\mathcal{D}^m+C\sqrt{\mathfrak{E}^m}\int_0^t\mathfrak{D}^{m+1}.\end{aligned}\tag{3.168}$$

As to the second term on RHS of (3.167), we first use the integration by parts, and then the standard L^2-L^∞ type estimates in analogy to (3.160). For any $0\le q\le k$ (recall $\partial^a_b=\partial^a_\theta\partial^b_t$):

$$\begin{aligned}&\Big|\int_0^t\int_{\mathbb{S}^1}\partial^2_qf^m\partial_{k+1-q}(\frac{1}{R^m|g^m|})[\gamma^m,\partial_{t^k}][u_n^{m+1}]^+_-\circ\phi^m|g^m|\Big|\le\Big|\int_0^t\int_{\mathbb{S}^1}\partial^1_qf^m\partial^1_{k+1-q}(\frac{1}{R^m|g^m|})[\gamma^m,\partial_{t^k}][u_n^{m+1}]^+_-\circ\phi^m|g^m|\Big|\\&+\Big|\int_0^t\int_{\mathbb{S}^1}\partial^1_qf^m\partial_{k+1-q}(\frac{1}{R^m|g^m|})\partial_\theta\big([\gamma^m,\partial_{t^k}][u_n^{m+1}]^+_-\circ\phi^m|g^m|\big)\Big|\le C\sup_{0\le s\le t}\mathfrak{E}^m\big(\int_0^t\mathcal{D}^m+\int_0^t\mathfrak{D}^{m+1}\big).\end{aligned}\tag{3.169}$$

where we used part (a) of Lemma 3.3 in the last estimate. The last term on RHS of (3.163) is estimated as follows:

$$\int_0^t\int_{\Gamma^m}V_{\Gamma^m}|\nabla u^{m+1,\pm}_{t^k}|^2\le C\sup_{0\le s\le t}||V_{\Gamma^m}||_{L^\infty(\Gamma^m)}\int_0^t||u^{m+1}_{t^k}||^2_{H^2(\Omega^m)}\le C\sqrt{\mathfrak{E}^m}\int_0^t\mathfrak{D}^{m+1}.\tag{3.170}$$

Estimating $\int_0^t\int_{\mathbb{S}^1}Q^m_k$. Recall that Q^m_k is defined by (3.103), where Q is given by (2.75) with (3.100). The first four terms on RHS of (2.75) are the cross-terms, first of which is estimated as follows:

$$\Big|\int_0^t\int_{\mathbb{S}^1}f^{m+1}_{t^{k+1}}(f^{m+1}_{t^k}-f^m_{t^k})\Big|\le\lambda\int_0^t||f^{m+1}_{t^{k+1}}||^2_{L^2}+\frac{\bar{C}}{\lambda}\int_0^t\big(||f^{m+1}_{t^k}||^2_{L^2}+||f^m_{t^k}||^2_{L^2}\big).\tag{3.171}$$

The second cross-term is estimated analogously:

$$\Big|\int_0^t\int_{\mathbb{S}^1}f^{m+1}_{\theta t^{k+1}}(f^{m+1}_{\theta t^k}-f^m_{\theta t^k})\Big|\le\lambda\int_0^t||f^{m+1}_{\theta t^{k+1}}||^2_{L^2}+\frac{\bar{C}}{\lambda}\int_0^t\big(||f^{m+1}_{\theta t^k}||^2_{L^2}+||f^m_{\theta t^k}||^2_{L^2}\big).\tag{3.172}$$

The third cross-term is estimated similarly:

$$\Big|\epsilon\int_0^t\int_{\mathbb{S}^1}f^{m+1}_{\theta^2t^{k+1}}(f^{m+1}_{\theta^2t^k}-f^m_{\theta^2t^k})\Big|\le\lambda\int_0^t\epsilon||f^{m+1}_{\theta^2t^{k+1}}||^2_{L^2}+\frac{\bar{C}}{\lambda}\int_0^t\big(\epsilon||f^{m+1}_{\theta^2t^k}||^2_{L^2}+\epsilon||f^m_{\theta^2t^k}||^2_{L^2}\big)\tag{3.173}$$

As to the fourth cross-term, in the same way we show

$$\Big|\epsilon\int_0^t\int_{\mathbb{S}^1} f^{m+1}_{\theta^3 t^{k+1}}(f^{m+1}_{\theta^3 t^k}-f^m_{\theta^3 t^k})\Big|\leq\lambda\int_0^t \epsilon||f^{m+1}_{\theta^3 t^{k+1}}||^2_{L^2}+\frac{\bar{C}}{\lambda}\int_0^t\big(\epsilon||f^{m+1}_{\theta^3 t^k}||^2_{L^2}+\epsilon||f^m_{\theta^3 t^k}||^2_{L^2}\big) \tag{3.174}$$

The fifth term on RHS of (2.75) is estimated as follows:

$$\Big|\int_0^t\int_{\mathbb{S}^1} f^m_{t^k}f^{m+1}_{t^{k+1}}f^m\Big|\leq \sup_{0\leq s\leq t}||f^m(s)||_{L^\infty}\int_0^t\big\{||f^{m+1}_{t^{k+1}}||^2_{L^2}+||f^m_{t^k}||^2_{L^2}\big\}\leq C\sup_{0\leq s\leq t}||f^m(s)||_{H^1}\Big(\int_0^t\mathcal{D}^{m+1}+\int_0^t\mathcal{D}^m\Big). \tag{3.175}$$

In the sixth term on RHS of (2.75) with (3.100) we first integrate by parts and then estimate in the same way as in (3.175):

$$\begin{aligned}&\Big|\int_0^t\int_{\mathbb{S}^1} f^m_{\theta^2 t^k}f^{m+1}_{t^{k+1}}\Big(\frac{1}{R^m|g^m|}-1\Big)\Big|\leq\Big|\int_0^t\int_{\mathbb{S}^1} f^m_{\theta t^k}f^{m+1}_{\theta t^{k+1}}\Big(\frac{1}{R^m|g^m|}-1\Big)\Big|+\Big|\int_0^t\int_{\mathbb{S}^1} f^m_{\theta t^k}f^{m+1}_{t^{k+1}}\Big(\frac{1}{R^m|g^m|}\Big)_\theta\Big|\\&\leq C\Big|\Big|\frac{1}{R^m|g^m|}-1\Big|\Big|_{L^\infty}\int_0^t\big\{||f^{m+1}_{\theta t^{k+1}}||^2_{L^2}+||f^m_{\theta t^k}||^2_{L^2}\big\}+C\Big|\Big|\Big(\frac{1}{R^m|g^m|}\Big)_\theta\Big|\Big|_{L^\infty}\int_0^t\big\{||f^{m+1}_{t^{k+1}}||^2_{L^2}+||f^m_{\theta t^k}||^2_{L^2}\big\}\\&\leq C\sup_{0\leq s\leq t}\big(||f^m(s)||_{H^3}+\sqrt{\mathfrak{E}^m(s)}\big)\Big(\int_0^t\mathcal{D}^{m+1}+\int_0^t\mathcal{D}^m\Big),\end{aligned} \tag{3.176}$$

where we observe that $||\frac{1}{R^m|g^m|}-1||_{L^\infty}$, $||(\frac{1}{R^m|g^m|})_\theta||_{L^\infty}\leq C||f^m||_{W^{2,\infty}}$.

Note that the seventh term of (2.75) with (3.100) is simply 0 and since $F\equiv 1$ and the eighth term on RHS of (2.75) is estimated in the same way as in (3.176). After integrating by parts, we obtain

$$\epsilon\Big|\int_0^t\int_{\mathbb{S}^1} f^m_{\theta^2 t^k}f^{m+1}_{\theta^4 t^{k+1}}\Big(\frac{1}{R^m|g^m|}-1\Big)\Big|\leq C\sup_{0\leq s\leq t}||f^m(s)||_{H^2}\Big(\int_0^t\hat{\mathcal{D}}^{m+1}+\int_0^t\hat{\mathcal{D}}^m\Big). \tag{3.177}$$

Note that next-to-last term of $\int_0^t\int_{\mathbb{S}^1}Q^m_k$, given by (3.103) and (2.75) (with (3.100) and (3.88)), involves the term $h^m_k=h^m_{k,1}+\epsilon h^m_{k,2}$, given by (3.96). Note that

$$\Big|\int_0^t\int_{\mathbb{S}^1} h^m_k\bar{u}^m_k\circ\phi^m|g^m|\Big|\leq\Big|\int_0^t\int_{\mathbb{S}^1} h^m_{k,1}\bar{u}^m_k\circ\phi^m|g^m|\Big|+\epsilon\Big|\int_0^t\int_{\mathbb{S}^1} h^m_{k,2}\bar{u}^m_k\circ\phi^m|g^m|\Big|. \tag{3.178}$$

Through the standard L^∞-L^2 type estimate and Sobolev inequality, we obtain

$$||h^m_{k,1}||_{L^2}\leq\sum_{i=0}^{k-1}||f^{m+1}_{t^{i+1}}||_{L^2}\Big|\Big|\Big(\frac{R^m}{|g^m|}\Big)_{t^{k-i}}\Big|\Big|_{L^\infty}\leq C\sqrt{\mathfrak{E}^m}\sqrt{\mathcal{D}^{m+1}}$$

where we used Lemma 3.3, part (c), in the second estimate. Observing that $\bar{u}^m_k=\partial^*_{t^k}u^{m+1}=u^{m+1}_{t^k}-[\gamma^m,\partial_{t^k}]u^{m+1}$ we estimate the first term on RHS of (3.178) as follows:

$$\begin{aligned}&\Big|\int_0^t\int_{\mathbb{S}^1} h^m_{k,1}\bar{u}^m_k\circ\phi^m|g^m|\Big|\leq\int_0^t||h^m_{k,1}||_{L^2}||\bar{u}^m_k||_{L^2(\Gamma^m)}\\&\leq\int_0^t||h^m_{k,1}||_{L^2}\big(||u^{m+1}_{t^k}||_{L^2(\Gamma^m)}+||[\gamma^m,\partial_{t^k}]u^{m+1}||_{L^2(\Gamma^m)}\big)\leq C\sqrt{\mathfrak{E}^m}\int_0^t\mathcal{D}^{m+1}+C\sqrt{\mathfrak{E}^m}\int_0^t\mathfrak{D}^{m+1},\end{aligned} \tag{3.179}$$

where part (a) of Lemma 3.3 has been used in the last inequality. As to the second term on RHS of (3.178), in the same way

$$\begin{aligned}&\epsilon\Big|\int_0^t\int_{\mathbb{S}^1} h^m_{k,2}\bar{u}^m_k\circ\phi^m|g^m|\Big|\leq C\epsilon\int_0^t||h^m_{k,2}||_{L^2(\mathbb{S}^1)}||u^{m+1}_{t^k}||_{L^2(\Gamma^m)}+\epsilon\Big|\int_0^t\int_{\mathbb{S}^1} h^m_{k,2}[\gamma^m,\partial_{t^k}]u^{m+1}\circ\phi^m|g^m|\Big|\\&\leq C\sup_{0\leq s\leq t}\hat{\mathfrak{E}}^m\int_0^t\hat{\mathcal{D}}^{m+1}+C\sup_{0\leq s\leq t}\mathfrak{E}^m\int_0^t\mathcal{D}^{m+1},\end{aligned} \tag{3.180}$$

where we used the inequality (3.161) in the last estimate. Thus, combining (3.178) - (3.180), we obtain

$$\Big|\int_0^t\int_{\mathbb{S}^1} h_k^m \bar{u}_k^m\circ\phi^m|g^m|\Big|\le C(\sup\sqrt{\mathfrak{E}_\epsilon^m}+\sup\sqrt{\mathcal{E}^m})\Big(\int_0^t\mathcal{D}^{m+1}\int_0^t\mathfrak{D}^{m+1}\Big). \tag{3.181}$$

The last term of $\int_0^t\int_{\mathbb{S}^1}Q_k^m$, given by (3.103) and (2.75), involves the term G_k^m, which is defined by (3.95) with (3.100). Using the standard L^∞-L^2 type estimate, Sobolev inequality and part (c) of Lemma 3.3, it is easy to show that

$$||G_k^m||_{L^2}\le C(||\mathbb{P}_1 f^m||_{L^\infty}+\sqrt{\mathfrak{E}^m})\sqrt{\mathcal{D}^m}. \tag{3.182}$$

Thus, by the Cauchy-Schwarz inequality, we obtain

$$\Big|\int_0^t\int_{\mathbb{S}^1}G_k^m f_{t^{k+1}}^{m+1}R^m\Big|\le C||G_k^m||_{L^2}||f_{t^{k+1}}^{m+1}||_{L^2}\le C(\sup_{0\le s\le t}||\mathbb{P}_1 f^m||_{L^\infty}+\sqrt{\mathfrak{E}^m})\Big(\int_0^t\mathcal{D}^m+\int_0^t\mathcal{D}^{m+1}\Big). \tag{3.183}$$

Furthermore, integrating by parts and applying the Cauchy-Schwarz inequality we obtain (recall $\partial_{k+1}^4=\partial_\theta^4\partial_t^{k+1}$):

$$\epsilon\Big|\int_0^t\int_{\mathbb{S}^1}G_k^m\partial_{k+1}^4 f^{m+1}\Big|\le\epsilon\Big|\int_0^t\int_{\mathbb{S}^1}(G_k^m)_\theta\partial_{k+1}^3 f^{m+1}\Big|\le\sqrt{\epsilon}||(G_k^m)_\theta||_{L^2}||\sqrt{\epsilon}\partial_{k+1}^3 f^{m+1}||_{L^2}. \tag{3.184}$$

Recalling the definition (3.95) of G_k^m, we have

$$\sqrt{\epsilon}||(G_k^m)_\theta||_{L^2}\le\sqrt{\epsilon}||N^*(f^m)_{\theta t^k}||_{L^2}+\sqrt{\epsilon}\sum_{q=0}^{k-1}\Big(||\partial_q^3 f^m\big(\frac{1}{R^m|g^m|}\big)_{t^{k-q}}||_{L^2}+||\partial_q^2 f^m\big(\frac{1}{R^m|g^m|}\big)_{\theta t^{k-q}}||_{L^2}\Big) \tag{3.185}$$

Using the Leibniz rule, L^∞-L^2 type estimates, it follows that ($N^*(f)$ is defined in the line after (3.96)):

$$\sqrt{\epsilon}||N^*(f^m)_{\theta t^k}||_{L^2}\le(||f^m||_{H^4}+\sqrt{\mathfrak{E}^m})(\hat{\mathfrak{D}}^m+\sqrt{\epsilon}\mathcal{D}^m). \tag{3.186}$$

Further, for any $0\le q\le k-1$

$$\sqrt{\epsilon}||\partial_q^3 f^m\big(\frac{1}{R^m|g^m|}\big)_{t^{k-q}}||_{L^2}\le C\sqrt{\epsilon}||\partial_q^3 f^m||_{L^2}||\big(\frac{1}{R^m|g^m|}\big)_{t^{k-q}}||_{H^1}\le C(\sqrt{\epsilon}||f^m||_{H^3}+\sqrt{\hat{\mathcal{D}}^m})\sqrt{\mathfrak{D}^m}. \tag{3.187}$$

Similarly, for any $0\le q\le k-1$

$$\begin{aligned}&\sqrt{\epsilon}\sum_{q=0}^{k-1}||\partial_q^2 f^m\big(\frac{1}{R^m|g^m|}\big)_{\theta t^{k-q}}||_{L^2}\le\sqrt{\epsilon}\sum_{q=0}^{k-1}||\partial_q^2 f^m||_{L^\infty}||\big(\frac{1}{R^m|g^m|}\big)_{\theta t^{k-q}}||_{L^2}\\&\le C\sum_{q=0}^{k/2}(||f^m||_{H^3}+\sqrt{\mathcal{D}^m})\sqrt{\hat{\mathcal{D}}^m}+C\sum_{q=k/2}^{k-1}(\sqrt{\epsilon}||f^m||_{H^3}+\sqrt{\hat{\mathcal{D}}^m})\sqrt{\mathfrak{D}^m}\end{aligned} \tag{3.188}$$

Hence, from (3.185) - (3.188), we obtain

$$\sqrt{\epsilon}||(G_k^m)_\theta||_{L^2}\le C(||f^m||_{H^4}+\sqrt{\mathfrak{E}^m})\big(\sqrt{\mathcal{D}^m}+\sqrt{\hat{\mathcal{D}}^m}+\sqrt{\mathfrak{D}^m}\big)+C\mathcal{D}^m+C\hat{\mathcal{D}}^m. \tag{3.189}$$

Combining (3.183) - (3.189), we conclude the bound on the last term of $\int_0^t\int_{\mathbb{S}^1}Q_k^m$:

$$\Big|\int_0^t\int_{\mathbb{S}^1}G_k^m\big(R^m f_{t^{k+1}}^{m+1}+\epsilon f_{\theta^4 t^{k+1}}^{m+1}\big)\Big|\le C(\sup_{0\le s\le t}||f^m||_{H^4}+\sqrt{\mathfrak{E}^m})\Big(\int_0^t\mathcal{D}_\epsilon^m+\int_0^t\mathcal{D}_\epsilon^{m+1}\Big). \tag{3.190}$$

Estimating $\int_0^t\int_{\mathbb{S}^1}S_k^m$. Recall that S_k^m is given by (3.103) and (2.76) (with (3.100)). The first four terms on RHS of (2.76) are the cross-terms:

$$\begin{aligned}&\Big|\int_0^t\int_{\mathbb{S}^1}f_{t^{k+1}}^{m+1}(f_{t^{k+1}}^{m+1}-f_{t^{k+1}}^m)\Big|\le\lambda\int_0^t||f_{t^{k+1}}^m||_{L^2}^2+\frac{\bar{C}}{\lambda}\int_0^t||f_{t^{k+1}}^{m+1}||_{L^2}^2\le\lambda\int_0^t\mathcal{D}^m+\frac{C}{\lambda}\int_0^t\big(\eta||u_{t^k}^{m+1}||_{H^2}^2+\frac{\bar{C}}{\eta}||u_{t^k}^{m+1}||_{H^1}^2\big)\\&\le\lambda\int_0^t\mathcal{D}^m+\frac{\bar{C}\eta}{\lambda}\int_0^t\mathfrak{D}^{m+1}+\frac{\bar{C}}{\eta\lambda}t\sup_{0\le s\le t}\mathcal{E}^{m+1},\end{aligned} \tag{3.191}$$

where we used the relation (1.3) and the trace inequality in the last inequality. Similarly, for the second term, we have

$$\begin{aligned}&\Big|\int_0^t\int_{\mathbb{S}^1}(f^m_{\theta t^{k+1}}-f^{m+1}_{\theta t^{k+1}})f^{m+1}_{\theta t^{k+1}}\Big|\leq\eta\int_0^t\int_{\mathbb{S}^1}|f^m_{\theta t^{k+1}}|^2+\frac{\bar{C}}{\eta}\int_0^t\int_{\mathbb{S}^1}|f^{m+1}_{\theta t^{k+1}}|^2\\&\leq\eta\int_0^t\mathcal{D}^m+\frac{\bar{C}\eta}{\lambda\epsilon}\int_0^t\int_\Omega|\nabla u^{m+1}_{t^k}|^2+\frac{\bar{C}}{\eta}\lambda\int_0^t\int_\Omega|\nabla^2u^{m+1}_{t^k}|^2.\end{aligned}\tag{3.192}$$

As to the third cross-term on RHS of (2.76), we have

$$\begin{aligned}&\epsilon\Big|\int_0^t\int_{\mathbb{S}^1}f^{m+1}_{\theta^2t^{k+1}}(f^{m+1}_{\theta^2t^{k+1}}-f^m_{\theta^2t^{k+1}})\Big|\leq\lambda\epsilon\int_0^t||f^m_{\theta^2t^{k+1}}||^2_{L^2}+\frac{\bar{C}\epsilon}{\lambda}\int_0^t||f^{m+1}_{\theta^2t^{k+1}}||^2_{L^2}\\&\leq\lambda\int_0^t\hat{\mathcal{D}}^m+\frac{\bar{C}}{\epsilon\lambda}\big(\eta||u^{m+1}_{t^k}||^2_{H^2}+\frac{C}{\eta}||u^{m+1}_{t^k}||^2_{H^1}\big)\leq\lambda\int_0^t\hat{\mathcal{D}}^m+\frac{\bar{C}\eta}{\epsilon\lambda}\int_0^t\mathfrak{D}^{m+1}+\frac{\bar{C}}{\epsilon\eta\lambda}t\sup_{0\leq s\leq t}\mathcal{E}^{m+1}.\end{aligned}\tag{3.193}$$

To estimate the fourth cross-term on RHS of (2.76), we apply the same estimates as in (3.193):

$$\epsilon\Big|\int_0^t\int_{\mathbb{S}^1}f^{m+1}_{\theta^3t^{k+1}}(f^{m+1}_{\theta^3t^{k+1}}-f^m_{\theta^3t^{k+1}})\Big|\leq\lambda\int_0^t\hat{\mathcal{D}}^m+\frac{\bar{C}\eta}{\epsilon\lambda}\int_0^t\mathfrak{D}^{m+1}+\frac{\bar{C}}{\epsilon\eta\lambda}t\sup_{0\leq s\leq t}\mathcal{E}^{m+1}.\tag{3.194}$$

In full analogy to the estimates (3.175) - (3.177), we can estimate the fifth, sixth, seventh and eighth term on RHS of (2.76) (note that the seventh term is actually 0, since $F\equiv1$). We obtain:

$$\begin{aligned}&\Big|\int_0^t\int_{\mathbb{S}^1}f^m_{t^{k+1}}f^{m+1}_{t^{k+1}}f^m\Big|+\Big|\int_0^t\int_{\mathbb{S}^1}f^m_{\theta^2t^{k+1}}f^{m+1}_{t^{k+1}}\big(\frac{1}{|g^m|}-1\big)\Big|+\epsilon\Big|\int_0^t\int_{\mathbb{S}^1}f^m_{\theta^2t^{k+1}}f^{m+1}_{\theta^4t^{k+1}}\big(\frac{1}{R^m|g^m|}-1\big)\Big|\\&\leq C\sup_{0\leq s\leq t}||f^m(s)||_{H^3}\big(\int_0^t\mathcal{D}^{m+1}_\epsilon+\int_0^t\mathcal{D}^m_\epsilon\big).\end{aligned}\tag{3.195}$$

The next-to-last term on RHS of $\int_0^t\int_{\mathbb{S}^1}S^m_k$ involves the expression G^m_k which is given by (3.95) and hence

$$(G^m_k)_t=-\sum_{q=0}^{k-1}f^m_{\theta^2t^{q+1}}\big(\frac{1}{R^m|g^m|}\big)_{t^{k-q}}-\sum_{q=0}^{k-1}f^m_{\theta^2t^q}\big(\frac{1}{R^m|g^m|}\big)_{t^{k-q+1}}+\partial_{t^{k+1}}N^*(f^m)$$

Separating the cases $q\leq k/2$ and $q>k/2$ and taking L^2 and L^∞-norms accordingly, we conclude

$$\int_0^t||(G^m_k)_t||^2_{L^2}\leq C(||\mathbb{P}_1f^m||^2_{L^2}+\sup_{0\leq s\leq t}\mathfrak{E}^m(s))\int_0^t\mathfrak{D}^m.\tag{3.196}$$

Hence

$$\begin{aligned}&\int_0^t\int_{\mathbb{S}^1}\Big(f^m_{\theta^2t^k}\big(\frac{1}{R^m|g^m|}\big)_t+(G^m_k)_t\Big)\partial^*_{t^k}u^{m+1}\circ\phi^m|g^m|\leq\lambda\int_0^t||u^{m+1}_{t^k}||^2_{L^2(\Gamma^m)}+\lambda\int_0^t||[\gamma^m,\partial_{t^k}]u^{m+1}||^2_{L^2(\Gamma^m)}\\&+\frac{C}{\lambda}\int_0^t||f^m_{\theta^2t^k}\big(\frac{1}{R^m|g^m|}\big)_t||^2_{L^2}+\frac{C}{\lambda}\int_0^t||(G^m_k)_t||^2_{L^2}\leq\lambda\int_0^t\mathcal{D}^{m+1}+\lambda\mathfrak{E}^m\int_0^t\mathfrak{D}^{m+1}+\frac{C}{\lambda}(||\mathbb{P}_1f^m||^2_{L^2}+\mathfrak{E}^m)\int_0^t\mathfrak{D}^m,\end{aligned}\tag{3.197}$$

where we used the trace inequality, part (a) of Lemma 3.3 and the estimate (3.196).

Let us introduce a small parameter θ_2 such that

$$\sup_{0\leq s\leq t}\big(\mathcal{E}(s)+\mathfrak{E}(s)\big)+\int_0^t\big\{\mathcal{D}(s)+\mathfrak{D}(s)\big\}\,ds\leq\theta_2.\tag{3.198}$$

We shall later express the desired constant θ_0 (from the statement of Lemma 3.2) with the help of θ_2. Using the identity (3.113) and estimates (3.144) - (3.150), (3.155), (3.162), (3.164) (3.170), (3.171) - (3.177),

(3.181), (3.190) and (3.191) - (3.176), we arrive at:

$$\begin{aligned}
&\mathcal{E}_\epsilon^{m+1}+\int_0^t \mathcal{D}_\epsilon^{m+1} \le \mathcal{E}_\epsilon(0)+\frac{C_0 t}{\lambda}\big(\mathcal{E}^m+\mathcal{E}^{m+1}+(\mathfrak{E}^{m+1})^2+C_A(\theta_1+\int_0^t \mathfrak{D}^m+\int_0^t \mathfrak{D}^{m+1})\big)\\
&+\big(\frac{\bar{C}t}{\eta\lambda}+\frac{\bar{C}t}{\epsilon\eta\lambda}\big)\mathcal{E}^{m+1}+\frac{\bar{C}t}{\lambda}\big(\mathcal{E}_\epsilon^m+\mathcal{E}_\epsilon^{m+1}\big)+C\big(\lambda+\sup_{0\le s\le t}||f^m||_{H^4}+\sqrt{\mathfrak{E}^m}+\mathfrak{E}^m\big)\big(\int_0^t \mathcal{D}_\epsilon^m+\int_0^t \mathcal{D}_\epsilon^{m+1}\big)\\
&+\big(C\lambda+\frac{\bar{C}\eta}{\lambda}+\frac{\bar{C}\eta}{\epsilon\lambda}+C\mathcal{E}_{(0)}^{1/2}\big)\int_0^t \mathfrak{D}^{m+1}+C\int_0^t \mathcal{D}^m \sup_{0\le s\le t}\mathfrak{E}^{m+1}(s)\\
&\le \mathcal{E}_\epsilon(0)+\frac{C_0 t}{\lambda}(\theta_1+\theta_2)+\frac{\bar{C}t}{\lambda}\theta_2+C\big(\lambda+\theta_2+\sqrt{\theta_2}\big)\theta_2+\frac{C_0 t}{\lambda}\big(\mathcal{E}^{m+1}+(\mathfrak{E}^{m+1})^2\big)\\
&+\big(\frac{\bar{C}t}{\eta\lambda}+\frac{\bar{C}t}{\epsilon\eta\lambda}\big)\mathcal{E}^{m+1}+\frac{\bar{C}t}{\lambda}\mathcal{E}_\epsilon^{m+1}+C\big(\lambda+\theta_2+\sqrt{\theta_2}\big)\int_0^t \mathcal{D}_\epsilon^{m+1}\\
&+\big(C\lambda+\frac{\bar{C}\eta}{\lambda}+\frac{\bar{C}\eta}{\epsilon\lambda}+C\theta_2^{1/2}+\frac{C_0 t}{\lambda}\big)\int_0^t \mathfrak{D}^{m+1}+C\theta_2\sup_{0\le s\le t}\mathfrak{E}^{m+1},
\end{aligned}\tag{3.199}$$

where we used the smallness assumption above. Let us set

$$\mathcal{K}(\theta_2,t):=\frac{C_0 t}{\lambda}(\theta_1+\theta_2)+\frac{\bar{C}t}{\lambda}\theta_2+C\big(\lambda+\theta_2+\sqrt{\theta_2}\big)\theta_2,\tag{3.200}$$

which shortens the notation for the constant term appearing on RHS of (3.199).

3.2.4 Reduction and representations of $\int_\Gamma v_{ss}v_n$ and $\int_\Gamma v_t v_n$

Just to provide intuition, assume for the moment that (u,f) is the solution to the Stefan problem (1.10) - (1.12). The energies $\mathfrak{E}$ and $\mathfrak{D}$ arise by applying the space-time differential operators to the heat equation (1.1). For instance, after applying $\partial_x\partial_t$ to the equation (1.1), multiplying by $-\Delta\partial_x\partial_t u$ and integrating over Ω, we run into the trouble of extracting the boundary energy contribution. The difficulty is contained in the restriction of the spatial differentiation operator ∂_{x^i} to the moving surface Γ. Namely $\partial_{x^i}=\tau_i\partial_s+n_i\partial_n$ on Γ, whereby $|n_i|$ is of order 1 and hence the normal component of ∂_{x^i} can not be assumed small. Thus, the restriction of the operator ∂^μ to Γ inherits non-trivial high order normal derivatives of u. This presents a difficulty in extracting the energy contribution, which is overcome through the *reduction* procedure explained in this subsection. The main idea is to recover the spatial derivatives via the equation in the bulk and the established temporal energy estimate in Subsection 3.2.3. We first decompose the Laplace operator on Γ as $\Delta u|_\Gamma=u_{ss}+\kappa u_n+u_{nn}$. Since $u_t=\Delta u$ in the bulk, we can represent u_{nn} in terms of the u_t (which is controlled by the temporal energy) and u_{ss} (which helps us extract the boundary energy contribution). We now rigorously justify the above heuristics.

In order to facilitate our arguments, we shall denote the multi-index μ of order $2p$, by an ordered $2p$-tuple $\mu=(i_1,i_2,\dots,i_{2p})$, where $i_k\in\{1,2\}$ for $k=1,\dots 2p$. Let $c=l-p-1$ and set $v=\partial^\mu u_{t^c}^{m+1}=\partial_{x^{i_1}}\partial_{x^{i_2}},\dots\partial_{x^{i_{2p}}}\partial_t^c u^{m+1}$. After applying the differential operator $\partial^\mu\partial_{t^c}$ to the equation (1.1) we obtain

$$v_t-\Delta v=0.\tag{3.201}$$

Multiplying the above equation by $-\Delta v$ and integrating over Ω^m, we get $\int_{\Omega^m}(\Delta v)^2-\int_{\Omega^m}v_t\Delta v=0$. On the other hand, using the Stokes theorem and denoting $\partial_{x^i}v=v_i$, we obtain

$$\int_{\Omega^m}(\Delta v)^2=\int_{\Omega^m}\nabla^2 v:\nabla^2 v-\int_{\Gamma^m\cup\partial\Omega}v_i v_{ij}n_j+\int_{\Gamma^m\cup\partial\Omega}\Delta v v_n.\tag{3.202}$$

Recall that $\partial_{x^i}=\tau_i\partial_s+n_i\partial_n$ on the moving boundary. In the following we employ the Einstein summation convention and sum over the repeated indices:

$$\begin{aligned}
&\int_{\Gamma^m\cup\partial\Omega}v_i v_{ij}n_j=\int_{\Gamma^m\cup\partial\Omega}\tau_i v_s v_{ij}n_j+\int_{\Gamma^m\cup\partial\Omega}n_i v_n v_{ij}n_j=\int_{\Gamma^m\cup\partial\Omega}v_s\tau_i(\tau_i\partial_s v_j+n_i\partial_n v_j)n_j+\int_{\Gamma^m\cup\partial\Omega}v_n v_{nn}\\
&=\int_{\Gamma^m\cup\partial\Omega}v_s\partial_s v_j n_j+\int_{\Gamma^m\cup\partial\Omega}v_n\big(\Delta v-\kappa v_n-v_{ss}\big)=\int_{\Gamma^m\cup\partial\Omega}v_s\big(\partial_s(v_j n_j)-v_j\partial_s(n_j)\big)+\int_{\Gamma^m\cup\partial\Omega}v_n\Delta v\\
&-\int_{\Gamma^m\cup\partial\Omega}\kappa v_n^2-\int_{\Gamma^m\cup\partial\Omega}v_n v_{ss}=\int_{\Gamma^m\cup\partial\Omega}v_s v_{ns}-\int_{\Gamma^m\cup\partial\Omega}\kappa v_s^2+\int_{\Gamma^m\cup\partial\Omega}v_n v_t\\
&-\int_{\Gamma^m\cup\partial\Omega}\kappa v_n^2-\int_{\Gamma^m\cup\partial\Omega}v_n v_{ss}=-2\int_{\Gamma^m\cup\partial\Omega}v_{ss}v_n-\int_{\Gamma^m\cup\partial\Omega}\big(\kappa v_s^2+\kappa v_n^2\big)+\int_{\Gamma^m\cup\partial\Omega}v_n v_t.
\end{aligned}$$

In the second equality we used the identity $n_i v_{ij} n_j = v_{nn}$. In the third equality we use $\tau_i\tau_i = |\tau|^2 = 1$ and $\Delta v = v_{ss} + \kappa v_n + v_{nn}$. In the fifth equality we use the fact that $\partial_s n_j = \kappa\tau_j$ together with $v_j\tau_j = v_s$. We also substitute v_t for Δv due to the equation (3.201). In the last equality we use integration by parts. Further note that

$$-\int_{\Omega^m} v_t\Delta v = \frac{1}{2}\partial_t\int_{\Omega^m}|\nabla v|^2 + \frac{1}{2}\int_{\Gamma^m} V_{\Gamma^m}|\nabla v|^2 - \int_{\Gamma^m\cup\partial\Omega} v_t v_n.$$

Hence, going back to (3.202) and the remark before it, we obtain

$$\int_{\Omega^m}\nabla^2 v:\nabla^2 v + \frac{1}{2}\partial_t\int_{\Omega^m}|\nabla v|^2 + \int_{\Gamma^m\cup\partial\Omega}\kappa(v_s^2+v_n^2) = \int_{\Gamma^m\cup\partial\Omega} v_t v_n - 2\int_{\Gamma^m\cup\partial\Omega} v_{ss}v_n - \frac{1}{2}\int_{\Gamma^m} V_{\Gamma^m}|\nabla v|^2$$

Upon integrating in time over the interval $[0,t]$, we arrive at the fundamental identity

$$\begin{aligned}&\frac{1}{2}\int_{\Omega^m}|\nabla v(t)|^2 + \int_0^t\int_{\Omega^m}|\nabla^2 v|^2 + \int_0^t\int_{\Gamma^m\cup\partial\Omega}\kappa(v_s^2+v_n^2)\\ &\quad = \frac{1}{2}\int_{\Omega^m}|\nabla v(0)|^2 - 2\int_0^t\int_{\Gamma^m\cup\partial\Omega} v_{ss}v_n + \int_0^t\int_{\Gamma^m\cup\partial\Omega} v_t v_n - \frac{1}{2}\int_0^t\int_{\Gamma^m} V_{\Gamma^m}|\nabla v|^2\end{aligned} \tag{3.203}$$

Convenient representation of v_{ss}, v_n and v_t. Motivated by the identity (3.203), our goal is to understand better the boundary integral terms $\int_{\Gamma^m\cup\partial\Omega} v_{ss}v_n$ and $\int_{\Gamma^m\cup\partial\Omega} v_t v_n$. To do so, we first proceed toward a description of terms v_{ss}, v_n and v_t in terms of tangential and time derivatives, as given by the formulas (3.211), (3.213) and (3.215).

Let $\mathcal{C}$ denote either Γ^m or $\partial\Omega$. Note that for $i = 1,2$, the operator ∂_{x^i} on $\mathcal{C}$ can be written as $\partial_{x^i} = \tau_i\partial_s + n_i\partial_n$. Hence, for a given multi-index $\mu = (i_1,\dots,i_{2p})$ of order $2p$, we have

$$\begin{aligned}\partial^\mu u_{t^c} &= (\tau_{i_1}\partial_s + n_{i_1}\partial_n)\dots(\tau_{i_{2p}}\partial_s + n_{i_{2p}}\partial_n)u_{t^c}\\ &= \sum_{k=0}^{2p}\sum_{\substack{J\subset\{i_1,\dots,i_{2p}\}\\ |J|=k}}\prod_{j\in J} n_j\prod_{j\in J^c}\tau_j\partial_{s^{2p-k}}\partial_{n^k}u_{t^c} + \sum_{j=1}^{2p}\sum_{a=0}^{2p-j}F_{a,j}(\tau,n)\partial_s^a\partial_n^{2p-a-j}u_{t^c}.\end{aligned} \tag{3.204}$$

Here $F_{a,j}(\tau,n)$ is given as a sum of the expressions of the form $C\partial_{s^{m(i_1)}}(\tau_{i_1}\vee n_{i_1}),\dots,\partial_{s^{m(i_{2p})}}(\tau_{i_{2p}}\vee n_{i_{2p}})$, where $m(i_k)\geq 0$ and $\sum_{k=1}^{2p} m(i_k) = j$ ($\partial_{s^{m(i)}}(\tau_i\vee n_i)$ equals either $\partial_{s^{m(i)}}\tau_i$ or $\partial_{s^{m(i)}}n_i$). Each of the summands in the second term on RHS of (3.204) is of lower order, and by this we mean that the number of derivatives hitting u_{t^c} equals at most $2p-1$. As to the first term of on RHS of (3.204), we carry out the crucial technical step, that we will refer to as the *reduction procedure*. Given $b\geq 2$ and $c\in\mathbb{N}$ we note the following:

$$\partial_n^b u_{t^c} = \partial_n^{b-2}\big(\Delta u_{t^c} - \kappa\partial_n u_{t^c} - \partial_{ss}u_{t^c}\big) = \partial_n^{b-2}u_{t^{c+1}} - \partial_{ss}\partial_n^{b-2}u_{t^c} - \kappa\partial_n^{b-1}u_{t^c}. \tag{3.205}$$

Carefully using (3.205), by induction we deduce for b even:

$$\partial_{n^b}u_{t^c} = \sum_{j=0}^{b/2}(-1)^m\binom{b/2}{m}\partial_s^{2m}u_{t^{c+b/2-m}} + \sum_{\alpha+\beta+2\gamma<b} M^b_{\alpha,\beta,\gamma}(\kappa)\partial_s^\alpha\partial_n^\beta u_{t^{c+\gamma}}; \tag{3.206}$$

and for b odd:

$$\partial_{n^b}u_{t^c} = \sum_{m=0}^{\lfloor b/2\rfloor}(-1)^m\binom{\lfloor b/2\rfloor}{m}\partial_s^{2m}\partial_n u_{t^{c+\lfloor b/2\rfloor-m}} + \sum_{\alpha+\beta+2\gamma<b} M^b_{\alpha,\beta,\gamma}(\kappa)\partial_s^\alpha\partial_n^\beta u_{t^{c+\gamma}}. \tag{3.207}$$

For a given triple (α,β,γ), the function $M^b_{\alpha,\beta,\gamma}(\kappa)$ is a sum of the expressions of the form $C\partial_{s^{i_1}}\kappa\dots\partial_{s^{i_m}}\kappa$, where $i_1,\dots,i_m\geq 0$ and $\sum|i_k| = \alpha'$ for some α' such that $\alpha'+\alpha\leq\lfloor\frac{2b}{3}\rfloor$. This is a counting argument and is easily seen by counting the occurrences of κ in the repeated application of the formula (3.205). To illustrate this, assume $b = 4$ and note that in the second iteration of (3.205), the term $\partial_{ss}\partial_n^{b-2}u_{t^c}$ generates the contribution $\partial_{ss}(\kappa\partial_n^{b-3}u_{t^c})$ and from the product rule this expression always has the form $\partial_s^{\alpha'}\kappa\partial_s^\alpha\partial_n^{b-3}u_{t^c}$, where

$\alpha+\alpha'\leq 2=\lfloor\frac{2b}{3}\rfloor$. We can argue inductively to conclude the claim. Using the relations (3.206) and (3.207), we may write for k even

$$\partial_{s^{2p-k}}\partial_{n^k}u_{t^c}=\sum_{m=0}^{k/2}(-1)^m\binom{k/2}{m}\partial_{s^{2p-k+2m}}u_{t^{c+b/2-m}}+\sum_{\alpha+\beta+2\gamma<2p}M^{1,k}_{\alpha,\beta,\gamma}(\tau,n)\partial_s^\alpha\partial_n^\beta u_{t^{c+\gamma}};\tag{3.208}$$

and for k odd:

$$\partial_{s^{2p-k}}\partial_{n^k}u_{t^c}=\sum_{m=0}^{\lfloor k/2\rfloor}(-1)^m\binom{\lfloor k/2\rfloor}{m}\partial_{s^{2p-k+2m}}\partial_n u_{t^{c+\lfloor b/2\rfloor-m}}+\sum_{\alpha+\beta+2\gamma<2p}M^{1,k}_{\alpha,\beta,\gamma}(\tau,n)\partial_s^\alpha\partial_n^\beta u_{t^{c+\gamma}}.\tag{3.209}$$

Here, $M^{1,k}_{\alpha,\beta,\gamma}(\tau,n)$ is defined analogously to $M^b_{\alpha,\beta,\gamma}(\kappa)$, whereby the total number α' of ∂_s derivatives in the summands of the form $\partial_{s^{i_1}}\kappa\ldots\partial_{s^{i_m}}\kappa$ satisfies $\alpha'+\alpha\leq 2p-k+\lfloor\frac{2k}{3}\rfloor$. Hence, using (3.204), (3.208) and (3.209), we obtain:

$$\begin{aligned}\partial^\mu u_{t^c}=&\sum_{k\,\mathrm{even}}\sum_{|J|=k}\prod_{\substack{j\in J\\ j'\in J^c}}n_j\tau_{j'}\sum_{m=0}^{k/2}(-1)^m\binom{k/2}{m}\partial_{s^{2p-k+2m}}u_{t^{c+k/2-m}}\\&+\sum_{k\,\mathrm{odd}}\sum_{|J|=k}\prod_{\substack{j\in J\\ j'\in J^c}}n_j\tau_{j'}\sum_{m=0}^{\lfloor k/2\rfloor}(-1)^j\binom{\lfloor k/2\rfloor}{m}\partial_{s^{2p-k+2m}}\partial_n u_{t^{c+\lfloor k/2\rfloor-m}}+\sum_{\alpha+\beta+2\gamma<2p}F_{\alpha,\beta,\gamma}(\tau,n)\partial_s^\alpha\partial_n^\beta u_{t^{c+\gamma}},\end{aligned}\tag{3.210}$$

where

$$F_{\alpha,\beta,\gamma}=\sum_{k=0}^{2p}\sum_{\substack{J\subset\{i_1,\ldots,i_{2p}\}\\|J|=k}}\prod_{j\in J}n_j\prod_{j\in J^c}\tau_j\,M^{1,k}_{\alpha,\beta,\gamma}+\sum_{j=1}^{2p}\sum_{a=0}^{2p-j}F_{a,j}(\tau,n)\partial_s^a\partial_n^{2p-a-j}u_{t^c}.$$

With the use of formula (3.210) we write out the expression for v_{ss} on Γ^m:

$$\begin{aligned}\partial_{ss}\partial^\mu u_{t^c}=&\sum_{k\,\mathrm{even}}\sum_{|J|=k}\prod_{\substack{j\in J\\ j'\in J^c}}n_j\tau_{j'}\sum_{m=0}^{k/2}(-1)^m\binom{k/2}{m}\partial_{s^{2p-k+2m+2}}u_{t^{c+k/2-m}}\\&+\sum_{k\,\mathrm{odd}}\sum_{|J|=k}\prod_{\substack{j\in J\\ j'\in J^c}}n_j\tau_{j'}\sum_{m=0}^{\lfloor k/2\rfloor}(-1)^m\binom{\lfloor k/2\rfloor}{m}\partial_{s^{2p-k+2m+2}}\partial_n u_{t^{c+\lfloor k/2\rfloor-m}}+\tilde F_\mu\\=:&\sum_{k\,\mathrm{even}}\sum_{|J|=k}\prod_{\substack{j\in J\\ j'\in J^c}}n_j\tau_{j'}\sum_{m=0}^{k/2}a_m+\sum_{k\,\mathrm{odd}}\sum_{|J|=k}\prod_{\substack{j\in J\\ j'\in J^c}}n_j\tau_{j'}\sum_{m=0}^{\lfloor k/2\rfloor}b_m+\tilde F_\mu,\end{aligned}\tag{3.211}$$

where we define $\tilde F_\mu:=\partial_{ss}\left\{\sum_{\alpha+\beta+2\gamma<2p}F_{\alpha,\beta,\gamma}(\tau,n)\partial_s^\alpha\partial_n^\beta u_{t^{c+\gamma}}\right\}$, and by product rule, we can write it in the form

$$\tilde F_\mu=\sum_{\alpha+\beta+2\gamma<2p+2}\tilde F_{\alpha,\beta,\gamma}(\tau,n)\partial_s^\alpha\partial_n^\beta u_{t^{c+\gamma}}.$$

Furthermore,

$$a_m=(-1)^m\binom{k/2}{m}\partial_{s^{2p-k+2m+2}}u_{t^{c+k/2-m}},\quad b_m=(-1)^m\binom{\lfloor k/2\rfloor}{m}\partial_{s^{2p-k+2m+2}}\partial_n u_{t^{c+\lfloor k/2\rfloor-m}}.\tag{3.212}$$

For v_n on Γ^m we obtain:

$$\begin{aligned}
\partial_n\partial^\mu u_{t^c} &= \sum_{k\,\mathrm{even}}\sum_{|J|=k}\prod_{\substack{j\in J\\ j'\in J^c}} n_j\tau_{j'}\sum_{m=0}^{k/2}(-1)^m\binom{k/2}{m}\partial_{s^{2p-k+2m}}\partial_n u_{t^{c+k/2-m}}\\
&+\sum_{k\,\mathrm{odd}}\sum_{|J|=k}\prod_{\substack{j\in J\\ j'\in J^c}} n_j\tau_{j'}\sum_{m=0}^{\lfloor k/2\rfloor}(-1)^m\binom{\lfloor k/2\rfloor}{m}\big\{\partial_{s^{2p-k+2m}}u_{t^{c+\lfloor k/2\rfloor-m+1}}-\partial_{s^{2p-k+2m+2}}u_{t^{c+\lfloor k/2\rfloor-m}}\big\}+\bar F_\mu\\
&=: \sum_{k\,\mathrm{even}}\sum_{|J|=k}\prod_{\substack{j\in J\\ j'\in J^c}} n_j\tau_{j'}\sum_{m=0}^{k/2}c_m+\sum_{k\,\mathrm{odd}}\sum_{|J|=k}\prod_{\substack{j\in J\\ j'\in J^c}} n_j\tau_{j'}\sum_{m=0}^{\lfloor k/2\rfloor}(d_m-e_m)+\bar F_\mu,
\end{aligned} \tag{3.213}$$

where

$$\begin{aligned}
&c_m=(-1)^m\binom{k/2}{m}\partial_{s^{2p-k+2m}}\partial_n u_{t^{c+k/2-m}},\quad d_m=(-1)^m\binom{\lfloor k/2\rfloor}{m}\partial_{s^{2p-k+2m}}u_{t^{c+\lfloor k/2\rfloor-m+1}},\\
&e_m=(-1)^m\binom{\lfloor k/2\rfloor}{m}\partial_{s^{2p-k+2m+2}}u_{t^{c+\lfloor k/2\rfloor-m}}.
\end{aligned} \tag{3.214}$$

Furthermore, $\bar F_\mu$ is defined by $\bar F_\mu=\sum_{\alpha+\beta+2\gamma<2p+1}\bar F_{\alpha,\beta,\gamma}(\tau,n)\partial_s^\alpha\partial_n^\beta u_{t^{c+\gamma}}$. Similarly, using (3.210), we obtain an expression for v_t on Γ^m:

$$\begin{aligned}
\partial^\mu u_{t^{c+1}} &= \sum_{k\,\mathrm{even}}\sum_{|J|=k}\prod_{\substack{j\in J\\ j'\in J^c}} n_j\tau_{j'}\sum_{m=0}^{k/2}(-1)^m\binom{k/2}{m}\partial_{s^{2p-k+2m}}u_{t^{c+k/2-m+1}}\\
&+\sum_{k\,\mathrm{odd}}\sum_{|J|=k}\prod_{\substack{j\in J\\ j'\in J^c}} n_j\tau_{j'}\sum_{m=0}^{\lfloor k/2\rfloor}(-1)^m\binom{\lfloor k/2\rfloor}{m}\partial_{s^{2p-k+2m}}\partial_n u_{t^{c+\lfloor k/2\rfloor-m+1}}+\sum_{\alpha+\beta+2\gamma<2p}F_{\alpha,\beta,\gamma}(\tau,n)\partial_s^\alpha\partial_n^\beta u_{t^{c+\gamma+1}}\\
&=: \sum_{k\,\mathrm{even}}\sum_{|J|=k}\prod_{\substack{j\in J\\ j'\in J^c}} n_j\tau_{j'}\sum_{m=0}^{k/2}A_m+\sum_{k\,\mathrm{odd}}\sum_{|J|=k}\prod_{\substack{j\in J\\ j'\in J^c}} n_j\tau_{j'}\sum_{m=0}^{\lfloor k/2\rfloor}B_m+\hat F_\mu,
\end{aligned} \tag{3.215}$$

where

$$A_m=(-1)^m\binom{k/2}{m}\partial_{s^{2p-k+2m}}u_{t^{c+k/2-m+1}},\quad B_m=(-1)^m\binom{\lfloor k/2\rfloor}{m}\partial_{s^{2p-k+2m}}\partial_n u_{t^{c+\lfloor k/2\rfloor-m+1}} \tag{3.216}$$

and $\hat F_\mu$ is of the form $\hat F_\mu=\sum_{\alpha+\beta+2\gamma<2p}F_{\alpha,\beta,\gamma}(\tau,n)\partial_s^\alpha\partial_n^\beta u_{t^{c+\gamma+1}}$. With formulas (3.211), (3.213) and (3.215) in hand, we can evaluate the the integrals $\int_{\Gamma^m}v_{ss}v_n$ and $\int_{\Gamma^m}v_tv_n$.

Evaluating the expression $\int_{\Gamma^m}v_{ss}v_n$. The goal is to arrive at the identity (3.223), which helps us extract the boundary energy contribution and isolate the 'error' terms. Using (3.211) and (3.213), we may write

$$\begin{aligned}
&\sum_{|\mu|=2p}\int_{\mathcal C}v_{ss}v_n=\sum_{|\mu|=2p}\int_{\mathcal C}\partial_{ss}\partial^\mu u_{t^c}\partial_n\partial^\mu u_{t^c}\\
&=\sum_{\substack{i_r=1,\\ r=1,\dots,2p}}^{2}\int_{\mathcal C}\Big(\sum_{k\,\mathrm{even}}\sum_{|J|=k}\prod_{\substack{j\in J\\ j'\in J^c}} n_j\tau_{j'}\sum_{m=0}^{k/2}a_m+\sum_{k\,\mathrm{odd}}\sum_{|\bar J|=k}\prod_{\substack{j\in \bar J\\ j'\in \bar J^c}} n_j\tau_{j'}\sum_{m=0}^{\lfloor k/2\rfloor}b_m\Big)\times\\
&\times\Big(\sum_{k\,\mathrm{even}}\sum_{|J|=k}\prod_{\substack{j\in J\\ j'\in J^c}} n_j\tau_{j'}\sum_{m=0}^{k/2}c_m+\sum_{k\,\mathrm{odd}}\sum_{|\bar J|=k}\prod_{\substack{j\in \bar J\\ j'\in \bar J^c}} n_j\tau_{j'}\sum_{m=0}^{\lfloor k/2\rfloor}(d_m-e_m)\Big)+\sum_{|\mu|=2p}\int_{\mathcal C}v_{ss}\bar F_\mu+\sum_{|\mu|=2p}\int_{\mathcal C}v_n\tilde F_\mu-\sum_{|\mu|=2p}\int_{\mathcal C}\tilde F_\mu\bar F_\mu.
\end{aligned} \tag{3.217}$$

Let us simplify the first integral on RHS of (3.204). We observe that if J and $\bar J$ are two subsets of the index set $I=\{i_1,\dots,i_{2p}\}$ such that $J\neq\bar J$, then there obviously exists at least one index i_δ, such that $i_\delta\in J\cap\bar J^c$ or

$i_\delta \in \bar{J} \cap J^c$. Without loss of generality assume $i_\delta \in J \cap \bar{J}^c$. In this case, we note immediately

$$\sum_{i_\delta=1}^{2} \prod_{j\in J, j'\in J^c} \bar{\nu}_j \tau_{j'} \prod_{r\in \bar{J}, r'\in \bar{J}^c} n_r \tau_{r'} = \sum_{i_\delta=1}^{2} n_{i_\delta} \tau_{i_\delta} \prod_{j\in J\setminus i_\delta, j'\in J^c} \bar{\nu}_j \tau_{j'} \prod_{r\in \bar{J}, r'\in \bar{J}^c \setminus i_\delta} n_r \tau_{r'} = 0,$$

since $\sum_{i_\delta=1}^{2} n_{i_\delta} \tau_{i_\delta} = n\cdot\tau = 0$. Because of this, the first term on RHS of (3.217) reads:

$$\begin{aligned}
&\int_{\mathcal{C}} \sum_{\substack{i_r=1\\ r\in\{1,\dots,2p\}}}^{2} \sum_{k \text{ even}} \sum_{\substack{|J|=k\\ J\subset I}} \prod_{j\in J, j'\in J^c} n_j^2 \tau_{j'}^2 \sum_{m=0}^{k/2} a_m \sum_{m=0}^{k/2} c_m + \int_{\mathcal{C}} \sum_{\substack{i_r=1\\ r\in\{1,\dots,2p\}}}^{2} \sum_{k \text{ odd}} \sum_{\substack{|J|=k\\ J\subset I}} \prod_{j\in J, j'\in J^c} n_j^2 \tau_{j'}^2 \sum_{m=0}^{\lfloor k/2\rfloor} b_m \sum_{m=0}^{\lfloor k/2\rfloor} (d_m - e_m)\\
&= \sum_{k \text{ even}} \binom{2p}{k} \int_{\mathcal{C}} \sum_{m=0}^{k/2} a_m \sum_{m=0}^{k/2} c_m + \sum_{k \text{ odd}} \binom{2p}{k} \int_{\mathcal{C}} \sum_{m=0}^{\lfloor k/2\rfloor} b_m \sum_{m=0}^{\lfloor k/2\rfloor} (d_m - e_m)
\end{aligned} \tag{3.218}$$

where we used $\sum_{j=1}^{2} \tau_j^2 = \sum_{j=1}^{2} n_j^2 = 1$ together with the fact that the number of k-subsets of a set of size $2p$ is $\binom{2p}{k}$. In the first term on the RHS of (3.218), we single out the $\int_{\mathcal{C}} a_{k/2} c_{k/2}$ contribution, as we shall use it to extract the leading order energy contribution (see (3.212) and (3.214)):

$$\begin{aligned}
&\sum_{k \text{ even}} \binom{2p}{k} \int_{\mathcal{C}} \sum_{m=0}^{k/2} a_m \sum_{m=0}^{k/2} c_m = \sum_{k \text{ even}} \binom{2p}{k} \int_{\mathcal{C}} a_{k/2} c_{k/2} + \sum_{k \text{ even}} \binom{2p}{k} \sum_{\mathcal{C}} (-a_{k/2} c_{k/2} + \sum_{m=0}^{k/2} a_m \sum_{m=0}^{k/2} c_m)\\
&= \sum_{k \text{ even}} \binom{2p}{k} \int_{\mathcal{C}} \partial_{s^{2p+2}} u_{t^c} \partial_{s^{2p}} \partial_n u_{t^c} + \sum_{k \text{ even}} \binom{2p}{k} \int_{\mathcal{C}} (-a_{k/2} c_{k/2} + \sum_{m=0}^{k/2} a_m \sum_{m=0}^{k/2} c_m).
\end{aligned} \tag{3.219}$$

Similarly, (see (3.212) and (3.214))

$$\begin{aligned}
&\sum_{k \text{ odd}} \binom{2p}{k} \int_{\mathcal{C}} \sum_{m=0}^{\lfloor k/2\rfloor} b_m \sum_{m=0}^{\lfloor k/2\rfloor} (d_m - e_m) = - \sum_{k \text{ odd}} \binom{2p}{k} \int_{\mathcal{C}} b_{\lfloor k/2\rfloor} e_{\lfloor k/2\rfloor}\\
&+ \sum_{k \text{ odd}} \binom{2p}{k} \int_{\mathcal{C}} \sum_{m=0}^{\lfloor k/2\rfloor} b_m \sum_{m=0}^{\lfloor k/2\rfloor} d_m - \sum_{k \text{ odd}} \binom{2p}{k} \int_{\mathcal{C}} (-b_{\lfloor k/2\rfloor} e_{\lfloor k/2\rfloor} + \sum_{m=0}^{\lfloor k/2\rfloor} b_m \sum_{m=0}^{\lfloor k/2\rfloor} e_m)\\
&= - \sum_{k \text{ even}} \binom{2p}{k} \int_{\mathcal{C}} \partial_{s^{2p+1}} u_{t^c} \partial_{s^{2p+1}} \partial_n u_{t^c} + \sum_{k \text{ odd}} \binom{2p}{k} \int_{\mathcal{C}} \sum_{m=0}^{\lfloor k/2\rfloor} b_m \sum_{m=0}^{\lfloor k/2\rfloor} d_m\\
&- \sum_{k \text{ odd}} \binom{2p}{k} \int_{\mathcal{C}} (-b_{\lfloor k/2\rfloor} e_{\lfloor k/2\rfloor} + \sum_{m=0}^{\lfloor k/2\rfloor} b_m \sum_{m=0}^{\lfloor k/2\rfloor} e_m)
\end{aligned} \tag{3.220}$$

Note that

$$\sum_{k \text{ even}} \binom{2p}{k} \int_{\mathcal{C}} \partial_{s^{2p+2}} u_{t^c} \partial_{s^{2p}} \partial_n u_{t^c} - \sum_{k \text{ odd}} \binom{2p}{k} \int_{\mathcal{C}} \partial_{s^{2p+1}} u_{t^c} \partial_{s^{2p+1}} \partial_n u_{t^c} = -2^{2p} \int_{\mathcal{C}} \partial_{s^{2p+1}} u_{t^c} \partial_{s^{2p+1}} \partial_n u_{t^c},$$

where we integrate by parts in the first integral on LHS and then use the binomial expansion formula. Hence, using this observation, we conclude from (3.217) - (3.220)

$$\sum_{|\mu|=2p} \int_{\mathcal{C}} v_{ss} v_n = -2^{2p} \int_{\mathcal{C}} \partial_{s^{2p+1}} u_{t^c} \partial_{s^{2p+1}} \partial_n u_{t^c} + \int_{\mathcal{C}} I_p + \sum_{|\mu|=2p} \int_{\mathcal{C}} v_{ss} \bar{F}_\mu + \sum_{|\mu|=2p} \int_{\mathcal{C}} v_n \tilde{F}_\mu - \sum_{|\mu|=2p} \tilde{F}_\mu \bar{F}_\mu, \tag{3.221}$$

where

$$\begin{aligned}
I_p &:= \sum_{k \text{ even}} \binom{2p}{k} \sum_{\mathcal{C}} (-a_{k/2} c_{k/2} + \sum_{m=0}^{k/2} a_m \sum_{m=0}^{k/2} c_m) + \sum_{k \text{ odd}} \binom{2p}{k} \int_{\mathcal{C}} \sum_{m=0}^{\lfloor k/2\rfloor} b_m \sum_{m=0}^{\lfloor k/2\rfloor} d_m\\
&- \sum_{k \text{ odd}} \binom{2p}{k} \int_{\mathcal{C}} (-b_{\lfloor k/2\rfloor} e_{\lfloor k/2\rfloor} + \sum_{m=0}^{\lfloor k/2\rfloor} b_m \sum_{m=0}^{\lfloor k/2\rfloor} e_m).
\end{aligned} \tag{3.222}$$

We use the trace formula (2.60) to further simplify the first integral on RHS of (3.221):

$$\begin{aligned}&\int_{\mathcal{C}}\partial_{s^{2p+1}}u_{t^c}\partial_{s^{2p+1}}\partial_n u_{t^c}=\int_{\mathcal{C}}\partial_{s^{2p+1}}\big(\partial_{t^c}^* u-[\gamma^m,\partial_{t^c}]u\big)\partial_{s^{2p+1}}\big(\partial_{t^c}^* u_n-[\gamma^m,\partial_{t^c}]u_n\big)\\
&=\int_{\mathcal{C}}\partial_{s^{2p+1}}\partial_{t^c}^* u\partial_{s^{2p+1}}\partial_{t^c}^* u_n-\int_{\mathcal{C}}\partial_{s^{2p+1}}[\gamma^m,\partial_{t^c}]u\partial_{s^{2p+1}}\partial_n u_{t^c}-\int_{\mathcal{C}}\partial_{s^{2p+1}}u_{t^c}\partial_{s^{2p+1}}[\gamma^m,\partial_{t^c}]u_n\\
&+\int_{\mathcal{C}}\partial_{s^{2p+1}}[\gamma^m,\partial_{t^c}]u\partial_{s^{2p+1}}[\gamma^m,\partial_{t^c}]u_n.\end{aligned}$$

Plugging the above in (3.221), we obtain the following important formula:

$$\begin{aligned}&\sum_{|\mu|=2p}\int_{\mathcal{C}}v_{ss}v_n=-2^{2p}\int_{\mathcal{C}}\partial_{s^{2p+1}}\partial_{t^c}^* u\partial_{s^{2p+1}}\partial_{t^c}^* u_n+2^{2p}\int_{\mathcal{C}}\partial_{s^{2p+1}}[\gamma^m,\partial_{t^c}]u\partial_{s^{2p+1}}\partial_n u_{t^c}\\
&+2^{2p}\int_{\mathcal{C}}\partial_{s^{2p+1}}u_{t^c}\partial_{s^{2p+1}}[\gamma^m,\partial_{t^c}]u_n-2^{2p}\int_{\mathcal{C}}\partial_{s^{2p+1}}[\gamma^m,\partial_{t^c}]u\partial_{s^{2p+1}}[\gamma^m,\partial_{t^c}]u_n\\
&+\int_{\mathcal{C}}I_p+\sum_{|\mu|=2p}\int_{\mathcal{C}}v_{ss}\bar{F}_\mu+\sum_{|\mu|=2p}\int_{\mathcal{C}}v_n\tilde{F}_\mu-\sum_{|\mu|=2p}\int_{\mathcal{C}}\tilde{F}_\mu\bar{F}_\mu.\end{aligned}\tag{3.223}$$

Evaluating the expression $\int_{\Gamma^m}v_tv_n$. Our goal is to prove the formula (3.227). Using the representations (3.213) and (3.215) of v_n and v_t respectively, just like in (3.217), (3.218) and (3.219), we arrive at:

$$\begin{aligned}&\sum_{|\mu|=2p}\int_{\mathcal{C}}v_tv_n=\sum_{k\,\text{even}}\binom{2p}{k}\int_{\mathcal{C}}\sum_{m=0}^{k/2}A_m\sum_{m=0}^{k/2}c_m+\sum_{k\,\text{odd}}\binom{2p}{k}\int_{\mathcal{C}}\sum_{m=0}^{\lfloor k/2\rfloor}B_m\sum_{m=0}^{\lfloor k/2\rfloor}(d_m-e_m)+\sum_{|\mu|=2p}\int_{\mathcal{C}}v_t\bar{F}_\mu\\
&+\sum_{|\mu|=2p}\int_{\mathcal{C}}v_n\hat{F}_\mu-\sum_{|\mu|=2p}\int_{\mathcal{C}}\hat{F}_\mu\bar{F}_\mu=\sum_{k\,\text{even}}\binom{2p}{k}\int_{\mathcal{C}}A_{k/2}c_{k/2}+\sum_{k\,\text{even}}\binom{2p}{k}\int_{\mathcal{C}}\Big(-A_{k/2}c_{k/2}+\sum_{m=0}^{k/2}A_m\sum_{m=0}^{k/2}c_m\Big)\\
&+\sum_{k\,\text{odd}}\binom{2p}{k}\int_{\mathcal{C}}\sum_{m=0}^{\lfloor k/2\rfloor}B_m\sum_{m=0}^{\lfloor k/2\rfloor}(d_m-e_m)+\sum_{|\mu|=2p}\int_{\mathcal{C}}v_t\bar{F}_\mu+\sum_{|\mu|=2p}\int_{\mathcal{C}}v_n\hat{F}_\mu-\sum_{|\mu|=2p}\int_{\mathcal{C}}\hat{F}_\mu\bar{F}_\mu.\end{aligned}\tag{3.224}$$

we only concentrate on the case when k is even, whereas the odd k-s are treated as an error term. Note that

$$\sum_{k\,\text{even}}\binom{2p}{k}\int_{\Gamma^m}A_{k/2}c_{k/2}=2^{2p-1}\int_{\mathcal{C}}\partial_{s^{2p}}u_{t^{c+1}}^{m+1}\partial_{s^{2p}}\partial_n u_{t^c}^{m+1}.\tag{3.225}$$

Let us abbreviate

$$J_p:=\sum_{k\,\text{even}}\binom{2p}{k}\int_{\mathcal{C}}\Big(-A_{k/2}c_{k/2}+\sum_{m=0}^{k/2}A_m\sum_{m=0}^{k/2}c_m\Big)+\sum_{k\,\text{odd}}\binom{2p}{k}\int_{\mathcal{C}}\sum_{m=0}^{\lfloor k/2\rfloor}B_m\sum_{m=0}^{\lfloor k/2\rfloor}(d_m-e_m).$$

Using (3.224) and (3.225), we arrive at the following identity:

$$\sum_{|\mu|=2p}\int_{\mathcal{C}}v_tv_n=2^{2p-1}\int_{\mathcal{C}}\partial_{s^{2p}}u_{t^{c+1}}^{m+1}\partial_{s^{2p}}\partial_n u_{t^c}^{m+1}+\int_{\mathcal{C}}J_p+\sum_{|\mu|=2p}\int_{\mathcal{C}}v_t\bar{F}_\mu+\sum_{|\mu|=2p}\int_{\mathcal{C}}v_n\hat{F}_\mu-\sum_{|\mu|=2p}\int_{\mathcal{C}}\hat{F}_\mu\bar{F}_\mu.\tag{3.226}$$

Just like in (3.223), we use the trace formula (2.60) to simplify the first integral on RHS of (3.226) and we obtain, in analogy to (3.223), an important identity:

$$\begin{aligned}&\sum_{|\mu|=2p}\int_{\mathcal{C}}v_tv_n=2^{2p-1}\int_{\mathcal{C}}\partial_{s^{2p}}\partial_{t^{c+1}}^* u^{m+1}\partial_{s^{2p}}\partial_{t^c}^* u_n^{m+1}-2^{2p-1}\int_{\mathcal{C}}\partial_{s^{2p}}[\gamma^m,\partial_{t^{c+1}}]u^{m+1}\partial_{s^{2p}}\partial_n u_{t^c}^{m+1}\\
&-2^{2p-1}\int_{\mathcal{C}}\partial_{s^{2p}}u_{t^{c+1}}^{m+1}\partial_{s^{2p}}[\gamma^m,\partial_{t^c}]u_n^{m+1}+2^{2p-1}\int_{\mathcal{C}}\partial_{s^{2p}}[\gamma^m,\partial_{t^{c+1}}]u^{m+1}\partial_{s^{2p}}[\gamma^m,\partial_{t^c}]u_n^{m+1}\\
&+\int_{\mathcal{C}}J_p+\sum_{|\mu|=2p}\int_{\mathcal{C}}v_t\bar{F}_\mu+\sum_{|\mu|=2p}\int_{\mathcal{C}}v_n\hat{F}_\mu-\sum_{|\mu|=2p}\int_{\mathcal{C}}\hat{F}_\mu\bar{F}_\mu.\end{aligned}\tag{3.227}$$

3.2.5 Space-time energy estimates

We proceed to estimate RHS of (3.223), term by term. We start with the last three integrals on RHS of (3.223):

Estimating terms $\int_{\mathcal{C}} v_{ss}\bar{F}_\mu$, $\int_{\mathcal{C}} v_n\tilde{F}_\mu$ and $\int_{\mathcal{C}} \tilde{F}_\mu\bar{F}_\mu$. Recall the definition (3.213) of $\bar{F}_\mu$ and the formula $\bar{F}_\mu = \sum_{\alpha+\beta+2\gamma<2p+1} \bar{F}_{\alpha,\beta,\gamma}(\tau,n)\partial_s^\alpha\partial_n^\beta u_{t^{c+\gamma}}$. Let us, thus, fix a triple (α,β,γ) of nonnegative integers satisfying $\alpha+\beta+2\gamma\le 2p$.

$$
\begin{aligned}
&\Big|\int_0^t\int_{\mathcal{C}} v_{ss}\bar{F}_{\alpha,\beta,\gamma}(\tau,n)\partial_s^\alpha\partial_n^\beta u_{t^{c+\gamma}}\Big| \le \int_0^t ||v_s||_{H^{1/2}(\mathcal{C})}||\bar{F}_{\alpha,\beta,\gamma}(\tau,n)\partial_s^\alpha\partial_n^\beta u^{m+1}_{t^{c+\gamma}}||_{H^{1/2}(\mathcal{C})}\\
&\le \eta\int_0^t ||v_s||^2_{H^{1/2}(\mathcal{C})} + \frac{C}{\eta}\int_0^t ||\bar{F}_{\alpha,\beta,\gamma}(\tau,n)\partial_s^\alpha\partial_n^\beta u^{m+1}_{t^{c+\gamma}}||^2_{H^{1/2}(\mathcal{C})} \le C\eta\int_0^t ||\nabla v||^2_{H^1(\Omega^m)} + \frac{C}{\eta}||\nabla u^{m+1}_{t^{c+\gamma}}||^2_{H^{\alpha+\beta}(\Omega^m)}\\
&\le C\eta\int_0^t ||\nabla v||^2_{H^1(\Omega^m)} + \frac{C\lambda}{\eta}||\nabla u^{m+1}_{t^{c+\gamma}}||^2_{H^{\alpha+\beta+1}(\Omega^m)} + \frac{C}{\lambda\eta}\int_0^t ||\nabla u^{m+1}_{t^{c+\gamma}}||^2_{L^2(\Omega^m)}\\
&\le C\eta\int_0^t ||\nabla v||^2_{H^1(\Omega^m)} + \frac{C\lambda}{\eta}\sum_{q=0}^p\int_0^t \mathfrak{D}_q^{m+1} + \frac{C}{\lambda\eta}\int_0^t \mathcal{D}^{m+1};
\end{aligned}
\tag{3.228}
$$

in the first inequality we use the $H^{1/2}$-version of the Cauchy-Schwarz inequality and in the second inequality we use simply the Cauchy-Schwarz inequality. In the third estimate we apply the trace inequality and in the fourth estimate we use the interpolation inequality between Sobolev spaces. In order for the fifth inequality to be valid, we have to make sure that (recall the definition (1.37))

$$
||\nabla u^{m+1}_{t^{c+\gamma}}||^2_{H^{\alpha+\beta+1}(\Omega^m)} \le C\sum_{q=0}^p\int_0^t \mathfrak{D}_q^{m+1} \tag{3.229}
$$

for our choice of the triple (α,β,γ). Note however, that $c+\gamma = l-1-(p-\gamma)$ and since $\alpha+\beta+2\gamma\le 2p$, we immediately conclude that $\alpha+\beta+1\le 2(p-\gamma)+1$. This implies that $||\nabla u^{m+1}_{t^{c+\gamma}}||^2_{H^{\alpha+\beta+1}(\Omega^m)} \le C\mathfrak{D}^{m+1}_{p-\gamma}$. Since $0\le\gamma\le p$, the inequality (3.229) follows. Thus, from (3.228) we infer

$$
\Big|\sum_{|\mu|=2p}\int_0^t\int_{\Gamma^m}\int_{\Gamma^m} v_{ss}\bar{F}_\mu\Big| \le \eta\int_0^t ||v||^2_{H^2(\Omega^m)} + \frac{C\lambda}{\eta}\sum_{q=0}^p\int_0^t \mathfrak{D}_q^{m+1} + \frac{C}{\lambda\eta}\int_0^t \mathcal{D}^{m+1}. \tag{3.230}
$$

Recall that $\tilde{F}_\mu = \sum_{\alpha+\beta+2\gamma<2p+2} \tilde{F}_{\alpha,\beta,\gamma}(\tau,n)\partial_s^\alpha\partial_n^\beta u_{t^{c+\gamma}}$. By virtue of the trace inequality, it is easy to see from the previous formula that

$$
||\tilde{F}_\mu||^2_{L^2(\mathcal{C})} \le C\sum_{q=0}^p\int_0^t \mathfrak{D}_q^{m+1}. \tag{3.231}
$$

The proof of this estimate is analogous to the proof of (3.229). Therefore

$$
\Big|\int_0^t\int_{\mathcal{C}} v_n\tilde{F}_\mu\Big| \le \eta\int_0^t ||\tilde{F}_\mu||^2_{L^2(\mathcal{C})} + \frac{C}{\eta}\int_0^t ||v_n||_{L^2(\mathcal{C})} \le C\eta\sum_{q=0}^p\int_0^t \mathfrak{D}_q^{m+1} + \frac{C\lambda}{\eta}\int_0^t ||\nabla^2 v||^2_{L^2(\Omega^m)} + \frac{C}{\lambda\eta}\int_0^t \mathcal{D}^{m+1}; \tag{3.232}
$$

where we used the estimate (3.231) in the second inequality and the trace inequality combined with the interpolation inequality between Sobolev spaces. Using the Cauchy-Schwarz inequality, the estimate (3.231) and the trace inequality, it is easy to obtain (recall that $\bar{F}_\mu$ is defined in the line after (3.214)):

$$
\Big|\int_0^t\int_{\mathcal{C}} \bar{F}_\mu\tilde{F}_\mu\Big| \le \lambda\sum_{q=0}^p\int_0^t \mathfrak{D}_q^{m+1} + \frac{C}{\lambda}\int_0^t \mathcal{D}^{m+1}. \tag{3.233}
$$

Estimating term $\int_0^t\int_{\mathcal{C}} I_p$. Recall the definition (3.222) of I_p and (3.212) and (3.214). Note that for $\mathcal{C}=\partial\Omega$, $\int_{\mathcal{C}} I_p = 0$, because in any of the products of the form a_mc_j, b_md_j and b_me_j at least one term has the form $\partial_{s^a}\partial_n u_{t^b}$ and is thus equal to 0 on $\partial\Omega$. Let us now assume that $\mathcal{C}=\Gamma^m$. Let $0\le k\le 2p$ be an even number and fix $0\le m,j\le k/2$ such that m or j is strictly smaller that $k/2$. Assume without loss of generality that

$j<k/2$. Let us denote $\alpha:=p-k/2+m$ and $\beta:=p-k/2+j$. Note that $\alpha\le p$ and $\beta\le p-1$. With these notations the expressions for a_m and c_j take the form $a_m=C\partial_{s^{2\alpha+2}}u^{\pm}_{t^{l-1-\alpha}}$ and $c_j=C\partial_{s^{2\beta}}\partial_n u^{\pm}_{t^{l-1-\beta}}$, where C is some positive constant. We can thus estimate

$$\begin{aligned}
&\Big|\int_0^t\int_{\Gamma^m} a_m c_j\Big| = C\Big|\int_0^t\int_{\Gamma^m}\partial_{s^{2\alpha+2}}u^{\pm}_{t^{l-1-\alpha}}\partial_{s^{2\beta}}\partial_n u^{\pm}_{t^{l-1-\beta}}\Big| \le C||\partial_{s^{2\alpha+1}}u^{\pm}_{t^{l-1-\alpha}}||_{H^{1/2}}||\partial_{s^{2\beta}}\partial_n u^{\pm}_{t^{l-1-\beta}}||_{H^{1/2}}\\
&\le \lambda\int_0^t ||\nabla u^{m+1}_{t^{l-1-\alpha}}||^2_{H^{2\alpha+1}(\Omega^m)}+\frac{C}{\lambda}\int_0^t||\nabla u_{t^{l-1-\beta}}||^2_{H^{2\beta+1}(\Omega^m)} \le \lambda\int_0^t\mathfrak{D}^{m+1}_{\alpha}+\frac{C}{\lambda}\int_0^t\mathfrak{D}^{m+1}_{\beta}\\
&\le \lambda\int_0^t\mathfrak{D}^{m+1}_p+\frac{C}{\lambda}\sum_{q=0}^{p-1}\int_0^t\mathfrak{D}^{m+1}_q.
\end{aligned}\tag{3.234}$$

In the first inequality we used the $H^{1/2}$ version of the Cauchy-Schwarz inequality. In the last inequality we made use of the fact that $\beta\le p-1$. In the case that k is odd, in analogous manner we can conclude

$$\Big|\int_0^t\int_{\Gamma^m} b_m d_j\Big| \le \lambda\int_0^t\mathfrak{D}^{m+1}_p+\frac{C}{\lambda}\sum_{q=0}^{p-1}\int_0^t\mathfrak{D}^{m+1}_q,\tag{3.235}$$

for a pair of indices $0\le m,j\le\lfloor\frac{k}{2}\rfloor$, such that m or j is strictly smaller than $\lfloor\frac{k}{2}\rfloor$. As to the expressions of the form $\int_0^t\int_{\Gamma^m}b_m e_j$, we have to allow for the possibility that both m and j may be equal to $\lfloor\frac{k}{2}\rfloor$. Let us abbreviate $\alpha:=p-\lfloor\frac{k}{2}\rfloor+m$ and $\beta:=p-\lfloor\frac{k}{2}\rfloor+j$. Note that $\alpha,\beta\le p$. With these notations the expressions for b_m and e_j take the form $b_m=C\partial_{s^{2\alpha+1}}u^{\pm}_{t^{l-1-\alpha}}$, and $c_j=C\partial_{s^{2(\beta-1)+1}}\partial_n u^{\pm}_{t^{l-1-(\beta-1)}}$, where C is some positive constant. Hence, using the same idea as in (3.234), we deduce

$$\Big|\int_0^t\int_{\Gamma^m} b_m e_j\Big| \le \lambda\int_0^t\mathfrak{D}^{m+1}_{\alpha}+\frac{C}{\lambda}\int_0^t\mathfrak{D}^{m+1}_{\beta-1}\le \lambda\int_0^t\mathfrak{D}^{m+1}_p+\frac{C}{\lambda}\sum_{q=0}^{p-1}\int_0^t\mathfrak{D}^{m+1}_q.\tag{3.236}$$

Here, we make use of the fact that $\beta\le p$ which implies $\beta-1\le p-1$. Combining (3.234) - (3.236), we conclude

$$\Big|\int_0^t\int_{\mathcal{C}} I_p\Big| \le \lambda\int_0^t\mathfrak{D}^{m+1}_p+\frac{C}{\lambda}\sum_{q=0}^{p-1}\int_0^t\mathfrak{D}^{m+1}_q.\tag{3.237}$$

We now proceed to estimate the second, third and the fourth term on RHS of (3.223).

Estimating term $\int_{\mathcal{C}}\partial_{s^{2p+1}}[\gamma,\partial_{t^c}]u^{m+1}\partial_{s^{2p+1}}\partial_n u^{m+1}_{t^c}$. From the definition (2.60) of $[\gamma,\partial_{t^c}]$, we realize that if $\mathcal{C}=\partial\Omega$ then $\int_{\mathcal{C}}\partial_{s^{2p+1}}[\gamma,\partial_{t^c}]u^{m+1}\partial_{s^{2p+1}}\partial_n u^{m+1}_{t^c}=0$. Hence, the expression we want to estimate is given by:

$$\sum_{d=0}^{c-1}\sum_{q+r=c-d}C_{d,q,r}\int_0^t\int_{\Gamma^m}\partial_{s^{2p+1}}\big(\partial^*_{t^{c-1-d}}(V^q_{\Gamma^m}(V_{\Gamma^m})^r_{\|})\partial_{s^r}\partial_{n^q}u^{m+1}_{t^d}\big)\partial_{s^{2p+1}}\partial_n u^{m+1}_{t^c}.$$

Let us fix some $0\le d\le c-1=l-p-2$. Then

$$\begin{aligned}
&\Big|\int_0^t\int_{\Gamma^m}\partial_{s^{2p+1}}\big(\partial^*_{t^{c-1-d}}(V^q_{\Gamma^m}(V_{\Gamma^m})^r_{\|})\partial_{s^r}\partial_{n^q}u^{m+1}_{t^d}\big)\partial_{s^{2p+1}}\partial_n u^{m+1}_{t^c}\Big|\\
&\le\int_0^t||\partial_{s^{2p+1}}\big(\partial^*_{t^{c-1-d}}(V^q_{\Gamma^m}(V_{\Gamma^m})^r_{\|})\partial_{s^r}\partial_{n^q}u^{m+1}_{t^d}\big)||_{H^{1/2}(\Omega^m)}||\partial_{s^{2p}}\partial_n u^{m+1}_{t^c}||_{H^{1/2}(\Omega^m)}\\
&\le\eta\int_0^t||\partial_{s^{2p}}\partial_n u^{m+1}_{t^c}||^2_{H^{1/2}(\Gamma^m)}+\frac{C}{\eta}\int_0^t||\partial^*_{t^{c-1-d}}(V^q_{\Gamma^m}(V_{\Gamma^m})^r_{\|})\partial_{s^r}\partial_{n^q}u^{m+1}_{t^d}||^2_{H^{2p+3/2}(\Gamma^m)},
\end{aligned}\tag{3.238}$$

where we used the $H^{1/2}$-version of the Cauchy-Schwarz inequality. Note that, by the trace inequality, we can estimate

$$\int_0^t||\partial_{s^{2p}}\partial_n u^{m+1}_{t^c}||^2_{H^{1/2}(\Gamma^m)}\le C\int_0^t||\nabla u^{m+1}_{t^c}||^2_{H^{2p+1}(\Omega^m)}\le C\int_0^t\mathfrak{D}^{m+1}_p.$$

To clarify this, note that

$$||\partial_{s^r}\partial_{n^q}u^{m+1}_{t^d}||^2_{H^{2p+3/2}(\Gamma^m)}\le C||\nabla u^{m+1}_{t^d}||^2_{H^{2p+1+q+r}(\Omega^m)},\tag{3.239}$$

where the trace and the interpolation inequalities are used respectively. By taking L^2 and L^∞ norms respectively and the trace inequality, we obtain (recall (1.34))

$$\int_0^t ||\partial^*_{t^{c-1-d}}(V^q_{\Gamma^m}(V_{\Gamma^m})^r_{\|})\partial_{s^r}\partial_{n^q}u^{m+1}_{t^d}||^2_{H^{2p+3/2}(\Gamma^m)} \le C\sup_{0\le s\le t}\mathfrak{E}^m(s)\sum_{q=p+1}^{l-1}\int_0^t \mathfrak{D}^{m+1}_q.$$

After writing $d=l-1-(l-1-d)$ and using the definition (1.37) of $\mathfrak{D}_q$, we need to establish $2p+1+q+r\le 2(l-1-d)+1$ in order to bound the first term on the right-most side of (3.239) by $\lambda\mathfrak{D}^{m+1}_{l-1-d}$. Since $q+r=c-d$ and $c=l-p-1$ it is a simple calculation to see that $2p+1+q+r\le 2(l-1-d)+1$ is equivalent to $d\le c$, which obviously holds. Using the inequalities between (3.238) and (3.239), we conclude

$$\Big|\int_{\mathcal{C}}\partial_{s^{2p+1}}[\gamma,\partial_{t^c}]u^{m+1}\partial_{s^{2p+1}}\partial_n u^{m+1}_{t^c}\Big| \le \eta C\int_0^t \mathfrak{D}^{m+1}_p + \frac{C}{\eta}\sup_{0\le s\le t}\mathfrak{E}^m(s)\sum_{q=p+1}^{l-1}\int_0^t\mathfrak{D}^{m+1}_q \tag{3.240}$$

In a fully analogous manner we can prove

$$\begin{aligned}&\Big|\int_{\Gamma^m}\partial_{s^{2p+1}}u^{m+1}_{t^c}\partial_{s^{2p+1}}[\gamma^m,\partial_{t^c}]u^{m+1}_n\Big| + \Big|\int_{\Gamma^m}\partial_{s^{2p+1}}[\gamma^m,\partial_{t^c}]u^{m+1}\partial_{s^{2p+1}}[\gamma^m,\partial_{t^c}]u^{m+1}_n\Big|\\ &\le \eta C\int_0^t\mathfrak{D}^{m+1}_p + \frac{C}{\eta}\sup_{0\le s\le t}\mathfrak{E}^m(s)\sum_{q=p+1}^{l-1}\int_0^t\mathfrak{D}^{m+1}_q.\end{aligned} \tag{3.241}$$

Evaluating $\int_{\mathcal{C}}\partial_{s^{2p+1}}\partial^*_{t^c}u\partial_{s^{2p+1}}\partial^*_{t^c}u_n$ via boundary conditions. Finally, we want to evaluate the first integral on RHS of (3.223) using the regularization (3.85) It is important to note that the pull-back of the operator ∂_s (on Γ^m, denoted by ∂_{Γ^m}) takes the form $\partial_{\Gamma^m}=\frac{1}{|g^m|}\partial_\theta$ on the sphere $\mathbb{S}^1$. Then the first integral on RHS of (3.223) takes the form (recall $H^m=H(g^m,R^m)$ where H is given by (1.8)):

$$\begin{aligned}&\int_0^t\int_{\Gamma^m}\partial_{s^{2p+1}}\partial^*_{t^c}u^{m+1}\partial_{s^{2p+1}}\partial^*_{t^c}[u^{m+1}_n]^+_- = \int_0^t\int_{\Gamma^m}\partial_{s^{2p+1}}\partial^*_{t^c}u^{m+1}\partial_{s^{2p+1}}\partial^*_{t^c}(V^{m+1}+\epsilon\Lambda^{m+1})\\ &=-\int_0^t\int_{\mathbb{S}^1}\partial^{2p+1}_{\Gamma^m}H^m_{t^c}\partial^{2p+1}_{\Gamma^m}\big(\frac{f^{m+1}_t R^m}{|g^m|}\big)_{t^c} - \epsilon\int_0^t\int_{\mathbb{S}^1}\partial^{2p+1}_{\Gamma^m}H^m_{t^c}\partial^{2p+1}_{\Gamma^m}\big(\frac{f^{m+1}_{\theta^4 t}}{|g^m|}\big)_{t^c}|g^m|.\end{aligned} \tag{3.242}$$

Let $h:\mathbb{S}^1\to\mathbb{R}$ be a smooth function and let the operator ∂_h be defined by $\partial_h=h\partial_\theta$. In addition to this let L be another smooth function, $L:\mathbb{S}^1\to\mathbb{R}$. It is easy to see that

$$\partial^k_h L = h^k L_{\theta^k} + \sum_{q=0}^{k-1}P_{k-q}(h)L_{\theta^q}, \tag{3.243}$$

where $P_{k-q}(h)=\sum_\nu c_\nu h_{\theta^{\nu_1}}\dots h_{\theta^{\nu_k}}$, where the summation runs over some multi-indices $\nu=(\nu_1,\dots,\nu_k)$, satisfying $\sum_{i=1}^k\nu_i=k-q$, $\nu_i\ge 0$. In this notation, we realize that $\partial_{\Gamma^m}=\partial_{|g^m|^{-1}}$ and we will make use of the formula (3.243) with $h=|g^m|^{-1}$. Recall the formula $H^m=-f^m-\frac{f^m_{\theta^2}}{R^m|g^m|}+N^*(f^m)$. Hence, using Leibniz' rule, we get

$$H^m_{t^c} = -f^m_{t^c}-\frac{f^m_{\theta^2 t^c}}{R^m|g^m|}-\sum_{q=0}^{c-1}f^m_{\theta^2 t^q}\big(\frac{1}{R^m|g^m|}\big)_{t^{c-q}}+N^*(f^m)_{t^c} = -f^m_{t^c}-\frac{f^m_{\theta^2 t^c}}{R^m|g^m|}-G^m_c, \tag{3.244}$$

where G^m_c is defined by (3.95). Hence from (3.244) and (3.243), with $h=|g^m|^{-1}$, we get

$$\begin{aligned}\partial^{2p+1}_{\Gamma^m}H^m_{t^c} &= \frac{1}{|g^m|^{2p+1}}H^m_{\theta^{2p+1}t^c}+\sum_{q=0}^{2p}P_{2p+1-q}\big(\frac{1}{|g^m|}\big)H^m_{\theta^q t^c} = -\frac{f^m_{\theta^{2p+1}t^c}}{|g^m|^{2p+1}}-\frac{f^m_{\theta^{2p+3}t^c}}{R^m|g^m|^{2p+2}}\\ &-\frac{1}{|g^m|^{2p+1}}\sum_{q=0}^{2p}f^m_{\theta^{q+2}t^c}\big(\frac{1}{R^m|g^m|}\big)_{\theta^{2p+1-q}}-\frac{1}{|g^m|^{2p+1}}\partial_{\theta^{2p+1}}G^m_c+\sum_{q=0}^{2p}P_{2p+1-q}\big(\frac{1}{|g^m|}\big)H^m_{\theta^q t^c}\\ &=-\frac{f^m_{\theta^{2p+1}t^c}}{|g^m|^{2p+1}}-\frac{f^m_{\theta^{2p+3}t^c}}{R^m|g^m|^{2p+2}}+G^{2p+1,c},\end{aligned} \tag{3.245}$$

where

$$G^{2p+1,c}:=-\frac{1}{|g^m|^{2p+1}}\sum_{q=0}^{2p}f^m_{\theta^{q+2}t^c}\big(\frac{1}{R^m|g^m|}\big)_{\theta^{2p+1-q}}-\frac{1}{|g^m|^{2p+1}}\partial_{\theta^{2p+1}}G^m_c+\sum_{q=0}^{2p}P_{2p+1-q}\big(\frac{1}{|g^m|}\big)H^m_{\theta^q t^c}. \quad (3.246)$$

Also note that

$$\partial_{t^c}\big((V^{m+1}+\Lambda^{m+1})\circ\phi^m\big)=-\partial_{t^c}\Big(\frac{f^{m+1}_t R^m}{|g^m|}+\epsilon\frac{f^{m+1}_{\theta^4 t}}{|g^m|}\Big)=-f^{m+1}_{t^{c+1}}\frac{R^m}{|g^m|}-\epsilon\frac{f^{m+1}_{t^{c+1}}}{|g^m|}+h^m_c \quad (3.247)$$

where h^m_c is defined by (3.96). Thus, from (3.247), (3.243) (with $h=|g^m|^{-1}$) and the Leibniz' rule, we obtain

$$\begin{aligned}
&-\partial^{2p+1}_{\Gamma^m}\Big(\frac{f^{m+1}_t R^m}{|g^m|}+\epsilon\frac{f^{m+1}_{\theta^4 t}}{|g^m|}\Big)_{t^c}=-\frac{1}{|g^m|^{2p+1}}\Big(\frac{f^{m+1}_t R^m}{|g^m|}+\epsilon\frac{f^{m+1}_{\theta^4 t}}{|g^m|}\Big)_{\theta^{2p+1}t^c}+\sum_{q=0}^{2p}P_{2p+1-q}\big(\frac{1}{|g^m|}\big)\Big((V^{m+1}+\Lambda^{m+1})\circ\phi^m\Big)_{\theta^q t^c}\\
&=\frac{1}{|g^m|^{2p+1}}\Big(-\frac{f^{m+1}_{t^{c+1}}R^m}{|g^m|}-\epsilon\frac{f^{m+1}_{\theta^4 t^{c+1}}}{|g^m|}+h^m_c\Big)_{\theta^{2p+1}}+\sum_{q=0}^{2p}P_{2p+1-q}\big(\frac{1}{|g^m|}\big)\Big((V^{m+1}+\Lambda^{m+1})\circ\phi^m\Big)_{\theta^q t^c}\\
&=-\frac{f^{m+1}_{\theta^{2p+1}t^{c+1}}R^m}{|g^m|^{2p+2}}-\epsilon\frac{f^{m+1}_{\theta^{2p+5}t^{c+1}}}{|g^m|^{2p+1}}-\frac{1}{|g^m|^{2p+1}}\sum_{q=0}^{2p}f^{m+1}_{\theta^q t^{c+1}}\big(\frac{R^m}{|g^m|}\big)_{\theta^{2p+1-q}}-\frac{\epsilon}{|g^m|^{2p+1}}\sum_{q=0}^{2p}f^{m+1}_{\theta^{q+4}t^{c+1}}\big(\frac{1}{|g^m|}\big)_{\theta^{2p+1-q}}\\
&+\frac{1}{|g^m|^{2p+1}}\partial_{\theta^{2p+1}}h^m_c+\sum_{q=0}^{2p}P_{2p+1-q}\big(\frac{1}{|g^m|}\big)\Big((V^{m+1}+\Lambda^{m+1})\circ\phi^m\Big)_{\theta^q t^c}\\
&=-\frac{f^{m+1}_{\theta^{2p+1}t^{c+1}}R^m}{|g^m|^{2p+2}}-\epsilon\frac{f^{m+1}_{\theta^{2p+5}t^{c+1}}}{|g^m|^{2p+1}}+h^{2p+1,c+1},
\end{aligned} \quad (3.248)$$

where

$$\begin{aligned}
h^{2p+1,c+1}:=&-\frac{1}{|g^m|^{2p+1}}\sum_{q=0}^{2p}f^{m+1}_{\theta^q t^{c+1}}\big(\frac{R^m}{|g^m|}\big)_{\theta^{2p+1-q}}-\frac{\epsilon}{|g^m|^{2p+1}}\sum_{q=0}^{2p}f^{m+1}_{\theta^{q+4}t^{c+1}}\big(\frac{1}{|g^m|}\big)_{\theta^{2p+1-q}}\\
&+\frac{1}{|g^m|^{2p+1}}\partial_{\theta^{2p+1}}h^m_c+\sum_{q=0}^{2p}P_{2p+1-q}\big(\frac{1}{|g^m|}\big)\Big((V^{m+1}+\Lambda^{m+1})\circ\phi^m\Big)_{\theta^q t^c}
\end{aligned} \quad (3.249)$$

Using formulas (3.244) and (3.247), we shall first evaluate the first integral on the rightmost side of (3.242):

$$\begin{aligned}
&\int_0^t\int_{\Gamma^m}\partial_{s^{2p+1}}\partial^*_{t^c}u^{m+1}\partial_{s^{2p+1}}\partial^*_{t^c}(V^{m+1}+\epsilon\Lambda^{m+1})=\int_0^t\int_{\mathbb{S}^1}\partial^{2p+1}_{\Gamma^m}H^m_{t^c}\partial^{2p+1}_{\Gamma^m}\big(\frac{f^{m+1}_t R^m}{|g^m|}+\epsilon\frac{f^{m+1}_{\theta^4 t}}{|g^m|}\big)_{t^c}|g^m|\\
&=\int_0^t\int_{\mathbb{S}^1}\Big\{\frac{f^m_{\theta^{2p+1}t^c}}{|g^m|^{2p+1}}+\frac{f^m_{\theta^{2p+3}t^c}}{R^m|g^m|^{2p+2}}-G^{2p+1,c}\Big\}\Big\{\frac{f^{m+1}_{\theta^{2p+1}t^{c+1}}R^m}{|g^m|^{2p+2}}+\epsilon\frac{f^{m+1}_{\theta^{2p+5}t^{c+1}}}{|g^m|^{2p+1}}-h^{2p+1,c+1}\Big\}|g^m|.
\end{aligned} \quad (3.250)$$

Now we may apply the identities (2.78) and (2.79) to the integral on the right-most side of (3.250). We set $\chi=f^m_{\theta^{2p+1}t^c}$, $\omega=f^{m+1}_{\theta^{2p+1}t^c}$, $F=|g^m|^{2p+1}$, $\psi=R^m$, $g=g^m$, $G=G^{2p+1,c}$ and $h=h^{2p+1,c+1}$. Thus, from (2.78) and (2.79), we obtain:

$$\begin{aligned}
&\int_0^t\int_{\Gamma^m}\partial_{s^{2p+1}}\partial^*_{t^c}u^{m+1}\partial_{s^{2p+1}}\partial^*_{t^c}(V^{m+1}+\epsilon\Lambda^{m+1})+\frac{1}{2}\int_{\mathbb{S}^1}\Big\{|f^{m+1}_{\theta^{2p+2}t^c}|^2-|f^{m+1}_{\theta^{2p+1}t^c}|^2\Big\}\Big|_0^t\\
&+\frac{\epsilon}{2}\int_{\mathbb{S}^1}\Big\{|f^{m+1}_{\theta^{2p+4}t^c}|^2-|f^{m+1}_{\theta^{2p+3}t^c}|^2\Big\}\Big|_0^t=\int_0^t\int_{\mathbb{S}^1}Q(f^m_{\theta^{2p+1}t^c},f^{m+1}_{\theta^{2p+1}t^c},R^m,|g^m|^{2p+1}),
\end{aligned} \quad (3.251)$$

where Q is defined by (2.75). For the sake of simplicity, let us denote $\tilde{Q}^m_{2p+1,c}:=Q(f^m_{\theta^{2p+1}t^c},f^{m+1}_{\theta^{2p+1}t^c},R^m,|g^m|^{2p+1})$. Recall the notation $\partial^a_b f:=f_{\theta^a t^b}$ for any $a,b\in\mathbb{N}$. Before we proceed with the estimates for $\int_0^t\int_{\mathbb{S}^1}\tilde{Q}^m_{2p+1,c}$ we state the following inequality that will be repeatedly used. Namely for any pair $(p,c)=(p,l-1-p)$, we have

$$\epsilon^2\int_0^t||\partial^{2p+4}_{c+1}f^{m+1}||^2_{L^2}\leq\eta\int_0^t\mathfrak{D}^{m+1}_p+\frac{C}{\eta}\int_0^t\mathfrak{E}^{m+1}_p \quad (3.252)$$

The estimate (3.252) follows by applying the operator ∂_c^{2p} to the regularized jump equation (3.85). We then take squares, integrate and use the trace inequality. We can further refine the estimate (3.252) using the interpolation on the term $\mathfrak{E}_p^{m+1}$ (recall (1.35)) to obtain

$$\epsilon^2\int_0^t ||\partial_{c+1}^{2p+4} f^{m+1}||_{L^2}^2 \leq \eta\int_0^t \mathfrak{D}_p^{m+1} + \frac{\tilde{C}t}{\eta}\sup_{0\leq s\leq t}\mathcal{E}^{m+1} \tag{3.253}$$

Estimating $\int_0^t\int_{\mathbb{S}^1}\tilde{Q}_{2p+1,c}^m$. The first four terms on RHS of (2.75) are the cross-terms. The generic constant used in the estimates for these terms will be denoted by $\tilde{C}$ to indicate that they are cross-terms. In the case of the first two cross-terms, we only analyze the case when $c>0$, because the term $\int_0^t\int_{\Gamma^m}\partial_{s^{2l-1}}u^{m+1}\partial_{s^{2l-1}}V^{m+1}$ is estimated separately. For the first term we have

$$\begin{aligned}&\Big|\int_0^t\int_{\mathbb{S}^1}\partial_{c+1}^{2p+1}f^{m+1}\big(\partial_c^{2p+1}f^{m+1}-\partial_c^{2p+1}f^m\big)\Big| \leq \lambda\int_0^t ||\partial_{c+1}^{2p+1}f^{m+1}||_{L^2}^2 + \frac{\tilde{C}}{\lambda}\int_0^t \big(||\partial_c^{2p+1}f^{m+1}||_{L^2}^2 + ||\partial_c^{2p+1}f^m||_{L^2}^2\big)\\ &\leq \lambda\int_0^t ||\partial_{c+1}^{2p+1}f^{m+1}||_{L^2}^2 + \frac{\tilde{C}t}{\lambda}\sup_{0\leq s\leq t}\big(||\partial_c^{2p+1}f^{m+1}||_{L^2}^2 + ||\partial_c^{2p+1}f^m||_{L^2}^2\big) \leq \lambda\int_0^t \mathfrak{D}_p^{m+1} + \frac{\tilde{C}t}{\lambda}\sup_{0\leq s\leq t}\big(\mathfrak{E}_p^{m+1}+\mathfrak{E}_p^m\big)\end{aligned} \tag{3.254}$$

As to the second term on RHS of (2.75), we have

$$\begin{aligned}&\Big|\int_0^t\int_{\mathbb{S}^1}\partial_{c+1}^{2p+2}f^{m+1}\big(\partial_c^{2p+2}f^{m+1}-\partial_c^{2p+2}f^m\big)\Big| \leq \lambda\int_0^t ||\partial_{c+1}^{2p+2}f^{m+1}||_{L^2}^2 + \frac{\tilde{C}t}{\lambda}\sup_{0\leq s\leq t}\big(||\partial_c^{2p+2}f^{m+1}||_{L^2}^2 + ||\partial_c^{2p+2}f^m||_{L^2}^2\big)\\ &\leq \lambda\int_0^t \mathfrak{D}_p^{m+1} + \frac{\tilde{C}\lambda t}{\epsilon^4}\sup_{0\leq s\leq t}\mathcal{E}^{m+1} + \frac{\tilde{C}t}{\lambda}\sup_{0\leq s\leq t}\big(\mathfrak{E}_p^{m+1}+\mathfrak{E}_p^m\big),\end{aligned} \tag{3.255}$$

where we argued analogously to the estimate (3.254) and in addition to that, in the last estimate we used the inequality (3.253) with $\eta=\epsilon^2$. The third and the fourth cross-terms are estimated in a fully analogous way, making use of (3.253) (with $\eta=\epsilon^2$) and the trace inequality:

$$\begin{aligned}&\epsilon\Big|\int_0^t\int_{\mathbb{S}^1}\partial_{c+1}^{2p+3}f^{m+1}\big(\partial_c^{2p+3}f^{m+1}-\partial_c^{2p+3}f^m\big)\Big| + \epsilon\Big|\int_0^t\int_{\mathbb{S}^1}\partial_{c+1}^{2p+4}f^{m+1}\big(\partial_c^{2p+4}f^{m+1}-\partial_c^{2p+4}f^m\big)\Big|\\ &\leq \lambda\int_0^t \mathfrak{D}_p^{m+1} + \frac{\tilde{C}\lambda t}{\epsilon^4}\sup_{0\leq s\leq t}\mathcal{E}^{m+1} + \frac{\tilde{C}t}{\lambda}\sup_{0\leq s\leq t}\big(\hat{\mathfrak{E}}_p^{m+1}+\hat{\mathfrak{E}}_p^m + \epsilon||\mathbb{P}_1 f^{m+1}||_{L^2}^2 + \epsilon||\mathbb{P}_1 f^m||_{L^2}^2\big)\\ &\leq \lambda\int_0^t \mathfrak{D}_p^{m+1} + \frac{\tilde{C}\lambda t}{\epsilon^4}\sup_{0\leq s\leq t}\mathcal{E}^{m+1} + \frac{\tilde{C}t}{\lambda}\sup_{0\leq s\leq t}\big(\hat{\mathfrak{E}}_p^{m+1}+\hat{\mathfrak{E}}_p^m\big) + \frac{\tilde{C}C_A t\epsilon}{\lambda}\big(\theta + \int_0^t \mathfrak{D}^m\big) + \frac{\tilde{C}C_A t\epsilon}{\lambda}\big(\theta_1 + t\int_0^t \mathfrak{D}^{m+1}\big).\end{aligned} \tag{3.256}$$

As to the fifth and the sixth term on RHS of (2.75), we first sum them and rewrite in a convenient way:

$$\begin{aligned}&-\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p+1}f^m\partial_{c+1}^{2p+1}f^{m+1}\big(\frac{R^m}{|g^m|^{4p+2}}-1\big) - \int_0^t\int_{\mathbb{S}^1}\partial_c^{2p+1}f_{\theta^2}^m\partial_{c+1}^{2p+1}f^{m+1}\big(\frac{1}{|g^m|^{4p+3}}-1\big)\\ &= -\int_0^t\int_{\mathbb{S}^1}\big(\partial_c^{2p+1}f_{\theta^2}^m + \partial_c^{2p+1}f^m\big)\partial_{c+1}^{2p+1}f^{m+1}\big(\frac{1}{|g^m|^{4p+3}}-1\big) - \int_0^t\int_{\mathbb{S}^1}\partial_c^{2p+1}f^m\partial_{c+1}^{2p+1}f^{m+1}\big(\frac{R^m}{|g^m|^{4p+2}}-\frac{1}{|g^m|^{4p+3}}\big).\end{aligned} \tag{3.257}$$

Notice that by the Sobolev inequality and the boundedness of $||f^m||_{H^2}$

$$\begin{aligned}&\sup_{0\leq s\leq t}||\frac{R^m}{|g^m|^{4p+2}}-\frac{1}{|g^m|^{4p+3}}||_{L^\infty} = \sup_{0\leq s\leq t}||\frac{1}{|g^m|^{4p+2}}\big(R^m-\frac{1}{|g^m|}\big)||_{L^\infty} \leq C||R^m-\frac{1}{|g^m|}||_{H^1}\\ &= C||2f^m + O((f^m)^2+(f_\theta^m)^2)||_{H^1} \leq C||f^m||_{H^2}.\end{aligned} \tag{3.258}$$

Then, keeping in mind that $c>0$, we easily estimate the second integral on RHS of (3.257):

$$\begin{aligned}&\Big|\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p+1}f^m\partial_{c+1}^{2p+1}f^{m+1}\big(\frac{R^m}{|g^m|^{4p+2}}-\frac{1}{|g^m|^{4p+3}}\big)\Big| \leq \sup_{0\leq s\leq t}||\frac{R^m}{|g^m|^{4p+2}}-\frac{1}{|g^m|^{4p+3}}||_{L^\infty}\\ &\times\big(\int_0^t ||\partial_c^{2p+1}f^m||_{L^2}^2 + \int_0^t ||\partial_{c+1}^{2p+1}f^{m+1}||_{L^2}^2\big) \leq C\sup_{0\leq s\leq t}||f^m||_{H^2}\big(\int_0^t \mathfrak{D}_p^m + \int_0^t \mathfrak{D}_p^{m+1}\big),\end{aligned} \tag{3.259}$$

where we recall that the bound on the first moments of $\partial_{c+1} f^{m+1}$: $||\partial_{c+1}\mathbb{P}_1 f^{m+1}||^2_{L^2} \leq C\mathfrak{D}^{m+1}_p$ holds due to (3.139). As to the first integral on RHS of (3.257), note that $\partial_c^{2p+1}(f^m_{\theta^2}+f^m)=\partial_c^{2p+1}(\mathbb{P}_{2+}f^m_{\theta^2}+\mathbb{P}_{2+}f^m)$ (recall (1.20)). Thus, we may write

$$-\int_0^t\int_{\mathbb{S}^1}(\partial_c^{2p+1}f^m_{\theta^2}+\partial_c^{2p+1}f^m)\partial_{c+1}^{2p+1}f^{m+1}\big(\frac{1}{|g^m|^{4p+3}}-1\big)=-\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p+1}\mathbb{P}_{2+}f^m_{\theta^2}\partial_{c+1}^{2p+1}f^{m+1}\big(\frac{1}{|g^m|^{4p+3}}-1\big)$$
$$-\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p+1}\mathbb{P}_{2+}f^m\partial_{c+1}^{2p+1}f^{m+1}\big(\frac{1}{|g^m|^{4p+3}}-1\big). \tag{3.260}$$

Just like in (3.259), we easily bound the second integral on RHS of (3.260):

$$\Big|\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p+1}\mathbb{P}_{2+}f^m\partial_{c+1}^{2p+1}f^{m+1}\big(\frac{1}{|g^m|^{4p+3}}-1\big)\Big|\leq C\sup_{0\leq s\leq t}||f^m||_{H^2}\big(\int_0^t\mathfrak{D}^m_p+\int_0^t\mathfrak{D}^{m+1}_p\big), \tag{3.261}$$

The term $|||g^m|^{4p+3}-1||_{L^\infty}$ is estimated analogously to (3.258). As to the first integral on RHS of (3.260), we first extract a full time derivative and a cross-term:

$$-\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p+1}\mathbb{P}_{2+}f^m_{\theta^2}\partial_{c+1}^{2p+1}f^{m+1}\big(\frac{1}{|g^m|^{4p+3}}-1\big)=-\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p+1}\mathbb{P}_{2+}f^m_{\theta^2}\partial_{c+1}^{2p+1}(\mathbb{P}_1f^{m+1}+\mathbb{P}_{2+}f^{m+1})\big(\frac{1}{|g^m|^{4p+3}}-1\big)$$
$$=\frac{1}{2}\partial_t\int_0^t\int_{\mathbb{S}^1}|\partial_c^{2p+2}\mathbb{P}_{2+}f^{m+1}|^2\big(\frac{1}{|g^m|^{4p+3}}-1\big)-\frac{1}{2}\int_0^t\int_{\mathbb{S}^1}|\partial_c^{2p+2}\mathbb{P}_{2+}f^{m+1}|^2\big(\frac{1}{|g^m|^{4p+3}}\big)_t$$
$$-\int_0^t\int_{\mathbb{S}^1}\partial_{c+1}^{2p+2}\mathbb{P}_{2+}f^{m+1}\big(\partial_c^{2p+2}\mathbb{P}_{2+}f^{m+1}-\partial_c^{2p+2}\mathbb{P}_{2+}f^m\big)\big(\frac{1}{|g^m|^{4p+3}}-1\big)$$
$$+\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p+1}\mathbb{P}_{2+}f^m_\theta\partial_{c+1}^{2p+1}\mathbb{P}_{2+}f^{m+1}\big(\frac{1}{|g^m|^{4p+3}}\big)_\theta-\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p+1}\mathbb{P}_{2+}f^m_{\theta^2}\partial_{c+1}^{2p+1}\mathbb{P}_1f^{m+1}\big(\frac{1}{|g^m|^{4p+3}}-1\big). \tag{3.262}$$

Observe that the first term on the right-most side of (3.262) gives exactly the last energy contribution in the definition (1.35) of $\mathfrak{E}_q$ with $q=p$. Furthermore,

$$\Big|\int_0^t\int_{\mathbb{S}^1}|\partial_c^{2p+2}\mathbb{P}_{2+}f^{m+1}|^2\big(\frac{1}{|g^m|^{4p+3}}\big)_t\Big|\leq\sup_{0\leq s\leq t}||\big(\frac{1}{|g^m|^{4p+3}}\big)_t||_{L^\infty}\int_0^t||\partial_c^{2p+2}\mathbb{P}_{2+}f^{m+1}||^2_{L^2}\leq C\sqrt{\sup\mathfrak{E}^m}\int_0^t\mathfrak{D}^{m+1}_{p+1}, \tag{3.263}$$

where we exploited the assumption $c>0$ in the estimate $\int_0^t||\partial_c^{2p+2}\mathbb{P}_{2+}f^{m+1}||^2_{L^2}\leq C\int_0^t\mathfrak{D}^{m+1}_{p+1}$. The third term on the right-most side of (3.262) is a cross-term and the estimates are analogous to (3.255):

$$\begin{aligned}&\Big|\int_0^t\int_{\mathbb{S}^1}\partial_{c+1}^{2p+2}\mathbb{P}_{2+}f^{m+1}\big(\partial_c^{2p+2}\mathbb{P}_{2+}f^{m+1}-\partial_c^{2p+2}\mathbb{P}_{2+}f^m\big)\big(\frac{1}{|g^m|^{4p+3}}-1\big)\Big|\\&\leq\lambda\int_0^t\mathfrak{D}^{m+1}_{p+1}+\frac{\tilde{C}\lambda t}{\epsilon^4}\sup_{0\leq s\leq t}\mathcal{E}^{m+1}+\frac{\tilde{C}t}{\lambda}\sup_{0\leq s\leq t}\big(\mathfrak{E}^{m+1}_p+\mathfrak{E}^m_p\big).\end{aligned} \tag{3.264}$$

The fourth term on the right-most side of (3.262) is estimated analogously to (3.259):

$$\begin{aligned}&\Big|\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p+1}\mathbb{P}_{2+}f^m_\theta\partial_{c+1}^{2p+1}\mathbb{P}_{2+}f^{m+1}\big(\frac{1}{|g^m|^{4p+3}}\big)_\theta\Big|\leq C||\big(\frac{1}{|g^m|^{4p+3}}\big)_\theta||_{L^\infty}\big(\int_0^t||\partial_c^{2p+1}\mathbb{P}_{2+}f^m_\theta||^2_{L^2}\\&+\int_0^t||\partial_{c+1}^{2p+1}\mathbb{P}_{2+}f^{m+1}||^2_{L^2}\big)\leq C\sup_{0\leq s\leq t}||f^m||_{H^2}\big(\int_0^t\mathfrak{D}^m_p+\int_0^t\mathfrak{D}^{m+1}_p\big).\end{aligned} \tag{3.265}$$

The last term on the right-most side of (3.262) is easily estimated after a routine application of integration by parts so that the number of θ derivatives hitting $\partial_c\mathbb{P}_{2+}f^m$ is exactly $2p+2$. We obtain

$$\Big|\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p+1}\mathbb{P}_{2+}f^m_{\theta^2}\partial_{c+1}^{2p+1}\mathbb{P}_1f^{m+1}\big(\frac{1}{|g^m|^{4p+3}}-1\big)\Big|\leq C\sup_{0\leq s\leq t}||\frac{1}{|g^m|^{4p+3}}-1||_{W^{1,\infty}}\big(\int_0^t||\partial_c^{2p+2}\mathbb{P}_{2+}f^m||^2_{L^2}$$
$$+\int_0^t||\partial_{c+1}\mathbb{P}_1f^{m+1}||^2_{L^2}\big)\leq C\sup_{0\leq s\leq t}||f^m||_{H^3}\big(\int_0^t\mathfrak{D}^m_p+\int_0^t\mathfrak{D}^{m+1}_p\big), \tag{3.266}$$

where the estimate analogous to (3.258) was used to get the bound $||\frac{1}{|g^m|^{4p+3}}-1||_{W^{1,\infty}}\leq C||f^m||_{H^3}$. Summing (3.259) - (3.266) and using (3.257), we conclude the bound on the sum of the fifth and the sixth term in the expression for $\int_0^t\int_{\mathbb{S}^1}\tilde{Q}^m_{2p+1,c}$:

$$\begin{aligned}
&\Big|-\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p+1}f^m\partial_{c+1}^{2p+1}f^{m+1}\big(\frac{R^m}{|g^m|^{4p+2}}-1\big)-\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p+1}f^m_{\theta^2}\partial_{c+1}^{2p+1}f^{m+1}\big(\frac{1}{|g^m|^{4p+3}}-1\big)\\
&-\frac{1}{2}\partial_t\int_0^t\int_{\mathbb{S}^1}|\partial_c^{2p+2}\mathbb{P}_{2+}f^{m+1}|^2\big(\frac{1}{|g^m|^{4p+3}}-1\big)\Big|\\
&\leq C\sup_{0\leq s\leq t}||f^m||_{H^2}\big(\int_0^t\mathfrak{D}_p^m+\int_0^t\mathfrak{D}_p^{m+1}\big)+C\sqrt{\sup\mathfrak{E}^m}\int_0^t\mathfrak{D}_{p+1}^{m+1}+\lambda\int_0^t\mathfrak{D}_{p+1}^{m+1}\\
&+\frac{\tilde{C}\lambda t}{\epsilon^4}\sup_{0\leq s\leq t}\mathcal{E}^{m+1}+\frac{\tilde{C}t}{\lambda}\sup_{0\leq s\leq t}\big(\mathfrak{E}_p^{m+1}+\mathfrak{E}_p^m\big)+C\sup_{0\leq s\leq t}||f^m||_{H^3}\big(\int_0^t\mathfrak{D}_p^m+\int_0^t\mathfrak{D}_p^{m+1}\big),\quad (c>0).
\end{aligned}\tag{3.267}$$

In order to estimate the expression $\int_0^t\int_{\Gamma^m}\partial_{s^{2p+1}}\partial_{t^c}^*u^{m+1}\partial_{s^{2p+1}}\partial_{t^c}^*V^{m+1}$ in the case $c=0$, we shall not evaluate it using the boundary conditions for u^{m+1}. Instead, we proceed in the following way (note that $c=0$ implies $p=l-1$):

$$\begin{aligned}
&\Big|\int_0^t\int_{\Gamma^m}\partial_{s^{2l-1}}u^{m+1}\partial_{s^{2l-1}}V^{m+1}\Big|\leq\lambda\int_0^t||\partial_{\Gamma^m}^{2l-1}\big(\frac{f_t^{m+1}R^m}{|g^m|}\big)||^2_{L^2}+\frac{C}{\lambda}\big(\frac{C}{\eta}\int_0^t||\nabla u^{m+1}||^2_{L^2(\Omega^m)}+\eta\int_0^t||\nabla u^{m+1}||^2_{H^{2l-1}(\Omega^m)}\big)\\
&\leq\lambda\int_0^t\mathfrak{D}_{l-1}^{m+1}+\frac{C}{\eta\lambda}\int_0^t\mathcal{D}^{m+1}+\frac{C\eta}{\lambda}\int_0^t\mathfrak{D}_{l-1}^{m+1},
\end{aligned}\tag{3.268}$$

where the trace inequality and the interpolation are used in the first inequality (note that the potentially large constant $\frac{C}{\eta\lambda}$ stands in front of the *temporal* dissipation term). We now estimate the seventh and the eighth term in the expression for $\int_0^t\int_{\mathbb{S}^1}\tilde{Q}^m_{2p+1,c}$ (where, recall, Q is given by (2.75)). Here, however, the possibility that $c=0$ is permitted and in some instances we have to proceed differently in our estimates if $c=0$. In order to estimate the seventh and the eighth term in the expression for $\int_0^t\int_{\mathbb{S}^1}\tilde{Q}^m_{2p+1,c}$ (where, recall, Q is given by (2.75)), we employ similar strategy as in the estimates leading to (3.267). In analogy to (3.257) we have:

$$\begin{aligned}
&\epsilon\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p+1}f^m\partial_c^{2p+1}f^{m+1}_{\theta^4t}\big(\frac{1}{|g^m|^{4p+2}}-1\big)+\epsilon\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p+1}f^m_{\theta^2}\partial_c^{2p+1}f^{m+1}_{\theta^4t}\big(\frac{1}{R^m|g^m|^{4p+3}}-1\big)\\
&=\epsilon\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p+1}(\mathbb{P}_{2+}f^m_{\theta^2}+\mathbb{P}_{2+}f^m)\partial_c^{2p+1}f^{m+1}_{\theta^4t}\big(\frac{1}{R^m|g^m|^{4p+3}}-1\big)\\
&+\epsilon\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p+1}f^m\partial_c^{2p+1}f^{m+1}_{\theta^4t}\big(\frac{1}{|g^m|^{4p+2}}-\frac{1}{R^m|g^m|^{4p+3}}\big).
\end{aligned}\tag{3.269}$$

Upon integrating by parts twice, the second term on RHS of (3.269) is estimated as follows:

$$\begin{aligned}
&\epsilon\Big|\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p+1}f^m\partial_c^{2p+1}f^{m+1}_{\theta^4t}\big(\frac{1}{|g^m|^{4p+2}}-\frac{1}{R^m|g^m|^{4p+3}}\big)\Big|\\
&\leq\epsilon\int_0^t||\partial_c^{2p+1}f^m||_{H^2}||\frac{1}{|g^m|^{4p+2}}-\frac{1}{R^m|g^m|^{4p+3}}||_{W^{2,\infty}}||\partial_c^{2p+1}f^{m+1}_{\theta^2t}||_{L^2}\leq\epsilon\sup_{0\leq s\leq t}||\partial_c^{2p+1}(\mathbb{P}_1f^m+\mathbb{P}_{2+}f^m)||^2_{H^2}||f^m||^2_{H^4}\\
&+C\epsilon t\int_0^t||\partial_c^{2p+1}f^{m+1}_{\theta^2t}||^2_{L^2}\leq C\sup_{0\leq s\leq t}||f^m||^2_{H^4}\hat{\mathfrak{E}}_p^m+C\epsilon||\partial_c\mathbb{P}_1f^m||^2_{L^2}(||\mathbb{P}_1f^m||^2_{L^2}+||\mathbb{P}_{2+}f^m||^2_{H^4})\\
&\leq Ct\int_0^t\hat{\mathfrak{D}}_p^{m+1}+C\sup_{0\leq s\leq t}||f^m||^2_{H^4}\hat{\mathfrak{E}}_p^m+\begin{cases}C\epsilon\sup_{0\leq s\leq t}||f^m||^2_{H^4}\mathcal{E}^m & c>0,\\ C\epsilon||\mathbb{P}_1f^m||^4_{L^2}+C||\mathbb{P}_1f^m||^2_{L^2}\hat{\mathfrak{E}}^m, & c=0.\end{cases}
\end{aligned}\tag{3.270}$$

To estimate the first integral on RHS of (3.269), in analogy to (3.270), we first note that

$$\begin{aligned}
&\epsilon\Big|\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p+1}\mathbb{P}_{2+}f^m\partial_c^{2p+1}f^{m+1}_{\theta^4t}\big(\frac{1}{R^m|g^m|^{4p+3}}-1\big)\Big|\leq\epsilon\int_0^t||\partial_c^{2p+1}\mathbb{P}_{2+}f^m||_{H^2}||\partial_c^{2p+1}f^{m+1}_{\theta^2t}||_{L^2}||\frac{1}{R^m|g^m|^{4p+3}}-1||_{W^{2,\infty}}\\
&\leq C\epsilon\sup_{0\leq s\leq t}||f^m||^2_{H^4}||\partial_c^{2p+1}\mathbb{P}_{2+}f^m||^2_{H^2}+Ct\epsilon\int_0^t||\partial_c^{2p+1}f^{m+1}_{\theta^2t}||^2_{L^2}\leq C\sup_{0\leq s\leq t}||f^m||^2_{H^4}\hat{\mathfrak{E}}_p^m+Ct\int_0^t\hat{\mathfrak{D}}_p^{m+1}
\end{aligned}\tag{3.271}$$

As to the first integral on RHS of (3.269), in analogy to (3.262), we note that

$$
\begin{aligned}
&-\epsilon\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p+1}\mathbb{P}_{2+}f^m_{\theta^2}\partial_c^{2p+1}f^{m+1}_{\theta^4t}\big(\frac{1}{R^m|g^m|^{4p+3}}-1\big)=-\epsilon\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p+1}\mathbb{P}_{2+}f^m_{\theta^2}\partial_c^{2p+1}(\mathbb{P}_1f^{m+1}_t+\mathbb{P}_{2+}f^{m+1}_{\theta^4t})\big(\frac{1}{R^m|g^m|^{4p+3}}-1\big)\\
&=\frac{1}{2}\epsilon\partial_t\int_0^t\int_{\mathbb{S}^1}|\partial_c^{2p+1}\mathbb{P}_{2+}f^{m+1}_{\theta^3}|^2\big(\frac{1}{R^m|g^m|^{4p+3}}-1\big)-\frac{1}{2}\epsilon\int_0^t\int_{\mathbb{S}^1}|\partial_c^{2p+1}\mathbb{P}_{2+}f^{m+1}_{\theta^3}|^2\big(\frac{1}{R^m|g^m|^{4p+3}}\big)_t\\
&-\epsilon\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p+1}\mathbb{P}_{2+}f^{m+1}_{\theta^3t}\big(\partial_c^{2p+1}\mathbb{P}_{2+}f^{m+1}_{\theta^3}-\partial_c^{2p+1}\mathbb{P}_{2+}f^m_{\theta^3}\big)\big(\frac{1}{R^m|g^m|^{4p+3}}-1\big)\\
&+\epsilon\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p+1}\mathbb{P}_{2+}f^m_{\theta^2}\partial_c^{2p+1}\mathbb{P}_{2+}f^{m+1}_{\theta^3t}\big(\frac{1}{R^m|g^m|^{4p+3}}\big)_\theta-\epsilon\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p+1}\mathbb{P}_{2+}f^m_{\theta^2}\partial_c^{2p+1}\mathbb{P}_1f^{m+1}_{\theta^4t}\big(\frac{1}{R^m|g^m|^{4p+3}}-1\big).
\end{aligned}
\tag{3.272}
$$

Observe that the first term on RHS of (3.272) gives exactly the last energy contribution in the definition (2.47) of $\hat{\mathfrak{E}}^\epsilon_q$. Routine estimates give the following bound:

$$
\begin{aligned}
&\Big|-\epsilon\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p+1}\mathbb{P}_{2+}f^m_{\theta^2}\partial_c^{2p+1}f^{m+1}_{\theta^4t}\big(\frac{1}{R^m|g^m|^{4p+3}}-1\big)-\frac{1}{2}\epsilon\partial_t\int_0^t\int_{\mathbb{S}^1}|\partial_c^{2p+1}\mathbb{P}_{2+}f^{m+1}_{\theta^3}|^2\big(\frac{1}{R^m|g^m|^{4p+3}}-1\big)\Big|\\
&\le\tilde{C}\sup_{0\le s\le t}||f^m||_{H^2}\big(\frac{t}{\epsilon^2}\sup_{0\le s\le t}\mathfrak{E}^{m+1}+\int_0^t\mathfrak{D}^{m+1}\big)+\tilde{C}t\sup_{0\le s\le t}||f^m||_{H^2}\sup_{0\le s\le t}\big(\mathfrak{E}^m_p+\mathfrak{E}^{m+1}_p\big)\\
&+\begin{cases}C\sup_{0\le s\le t}\sqrt{\mathfrak{E}^m(s)}\int_0^t\hat{\mathfrak{D}}^{m+1}_p, & c>0,\\ Ct\sup_{0\le s\le t}\sqrt{\mathfrak{E}^m(s)}\hat{\mathfrak{E}}^{m+1}_p, & c=0.\end{cases}
\end{aligned}
\tag{3.273}
$$

Note that we made use of the estimate (3.252) (with $\eta=\epsilon$), to get the bound $\epsilon\int_0^t||\partial_{c+1}^{2p+1}f^{m+1}_{\theta^3}||^2_{L^2}\le\frac{Ct}{\epsilon^2}\mathfrak{E}^{m+1}_p+\int_0^t\mathfrak{D}^{m+1}_p$. for any $p=0,\ldots,l-1$. From (3.269) - (3.273), we finally obtain

$$
\begin{aligned}
&\Big|\epsilon\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p+1}f^m\partial_c^{2p+1}f^{m+1}_{\theta^4t}\big(\frac{1}{|g^m|^{4p+2}}-1\big)+\epsilon\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p+1}f^m_{\theta^2}\partial_c^{2p+1}f^{m+1}_{\theta^4t}\big(\frac{1}{R^m|g^m|^{4p+3}}-1\big)\Big|\\
&\le Ct\int_0^t\hat{\mathfrak{D}}^{m+1}_p+C\sup_{0\le s\le t}||f^m||^2_{H^4}\hat{\mathfrak{E}}^m_p+\begin{cases}C\epsilon\sup_{0\le s\le t}||f^m||^2_{H^4}\mathcal{E}^m & c>0\\ C\epsilon||\mathbb{P}_1f^m||^4_{L^2}+C||\mathbb{P}_1f^m||^2_{L^2}\hat{\mathfrak{E}}^m, & c=0\end{cases}\\
&+C\sup_{0\le s\le t}||f^m||^2_{H^4}\hat{\mathfrak{E}}^m_p+Ct\int_0^t\hat{\mathfrak{D}}^{m+1}_p+\tilde{C}\sup_{0\le s\le t}||f^m||_{H^2}\big(\frac{Ct}{\epsilon^2}\mathfrak{E}^{m+1}_p+\int_0^t\mathfrak{D}^{m+1}_p\big)+\tilde{C}t\sup_{0\le s\le t}||f^m||_{H^2}\big(\mathfrak{E}^m_p+\mathfrak{E}^{m+1}_p\big)\\
&+\begin{cases}C\sup_{0\le s\le t}\sqrt{\mathfrak{E}^m(s)}\int_0^t\hat{\mathfrak{D}}^{m+1}_p, & c>0\\ Ct\sup\limits_{0\le s\le t}\sqrt{\mathfrak{E}^m(s)}\hat{\mathfrak{E}}^{m+1}_p, & c=0.\end{cases}
\end{aligned}
\tag{3.274}
$$

The last two terms in the expression for $\int_0^t\int_{\mathbb{S}^1}\tilde{Q}^m_{2p+1,c}$ involve the expressions $G^{2p+1,c}$ and $h^{2p+1,c+1}$, defined by (3.246) and (3.249) respectively. The following inequality holds

$$
\begin{aligned}
&\Big|\int_0^t\int_{\mathbb{S}^1}\Big(\frac{f^m_{\theta^{2p+1}t^c}}{|g^m|^{2p+1}}+\frac{f^m_{\theta^{2p+3}t^c}}{R^m|g^m|^{2p+2}}\Big)h^{2p+1,c+1}\Big|+\Big|\int_0^t\int_{\mathbb{S}^1}\Big(\frac{f^{m+1}_{\theta^{2p+1}t^{c+1}}R^m}{|g^m|^{2p+2}}+\epsilon\frac{f^{m+1}_{\theta^{2p+5}t^{c+1}}}{|g^m|^{2p+1}}\Big)G^{2p+1,c}\Big|\\
&+\Big|\int_0^t\int_{\mathbb{S}^1}h^{2p+1,c+1}G^{2p+1,c}\Big|\le\begin{cases}C(\sqrt{\mathfrak{E}^m}+||f^m||_{H^l})\big(\int_0^t\mathfrak{D}^m+\int_0^t\hat{\mathfrak{D}}^m\big)+C(\sqrt{\mathfrak{E}^m}+||f^m||_{H^l})\big(\int_0^t\mathfrak{D}^{m+1}+\int_0^t\hat{\mathfrak{D}}^{m+1}\big), & c>0,\\ Ct\sqrt{\mathfrak{E}^m}\hat{\mathfrak{E}}^m+Ct\sqrt{\mathfrak{E}^m}\hat{\mathfrak{E}}^{m+1}, & c=0;\end{cases}
\end{aligned}
\tag{3.275}
$$

where in the case $c=0$, we only estimate the ϵ-dependent terms on LHS. These estimates follow by a routine use of L^2 and L^∞ norms, together with the Cauchy-Schwarz inequality.

Reasoning in the way similar to the derivation of estimates (3.230), (3.232), (3.233), (3.237), (3.240)

and (3.241), we can deduce the bound on the last seven terms on RHS of (3.227):

$$\begin{aligned}&\int_0^t\Big|\int_{\mathcal{C}}\partial_{s^{2p}}[\gamma^m,\partial_{t^{c+1}}]u^{m+1}\partial_{s^{2p}}\partial_n u^{m+1}_{t^c}\Big|+\int_0^t\Big|\int_{\mathcal{C}}\partial_{s^{2p}}u^{m+1}_{t^{c+1}}\partial_{s^{2p}}[\gamma^m,\partial_{t^c}]u^{m+1}_n\Big|\\&+\int_0^t\Big|\int_{\mathcal{C}}\partial_{s^{2p}}[\gamma^m,\partial_{t^{c+1}}]u^{m+1}\partial_{s^{2p}}[\gamma^m,\partial_{t^c}]u^{m+1}_n\Big|+\int_0^t\Big|\int_{\mathcal{C}}J_p\Big|+\int_0^t\Big|\sum_{|\mu|=2p}\int_{\mathcal{C}}v_t\bar F_\mu\Big|\\&+\int_0^t\Big|\sum_{|\mu|=2p}\int_{\mathcal{C}}v_n\hat F_\mu\Big|+\int_0^t\Big|\sum_{|\mu|=2p}\int_{\mathcal{C}}\hat F_\mu\bar F_\mu\Big|\le\lambda\int_0^t\mathfrak{D}^{m+1}_p+\frac{C}{\lambda}\sum_{q=0}^{p-1}\int_0^t\mathfrak{D}^{m+1}_q+\frac{C}{\lambda}\int_0^t\mathcal{D}^{m+1}\\&+\lambda\int_0^t||\nabla^2v||^2_{L^2(\Omega)}+C\sup_{0\le s\le t}\mathfrak{E}^m(s)\sum_{q=p+1}^{l-1}\int_0^t\mathfrak{D}^{m+1}_q.\end{aligned}\tag{3.276}$$

Estimating RHS of (3.227). We now extract the leading order contribution from the first term on RHS of (3.227). Note that

$$\begin{aligned}&\int_0^t\int_{\Gamma^m}\partial_{s^{2p}}\partial^*_{t^{c+1}}u^{m+1}\partial_{s^{2p}}\partial^*_{t^c}[u^{m+1}_n]^+_-=\int_0^t\int_{\Gamma^m}\partial_{s^{2p}}\partial^*_{t^{c+1}}u^{m+1}\partial_{s^{2p}}\partial^*_{t^c}(V^{m+1}+\epsilon\Lambda^{m+1})\\&=\int_0^t\int_{\mathbb{S}^1}\Big\{\frac{f^m_{\theta^{2p}t^{c+1}}}{|g^m|^{2p}}+\frac{f^m_{\theta^{2p+2}t^{c+1}}}{R^m|g^m|^{2p+1}}-G^{2p,c+1}\Big\}\Big\{\frac{f^{m+1}_{\theta^{2p}t^{c+1}}R^m}{|g^m|^{2p+1}}+\epsilon\frac{f^{m+1}_{\theta^{2p+4}t^{c+1}}}{|g^m|^{2p}}-h^{2p,c+1}\Big\}|g^m|.\end{aligned}\tag{3.277}$$

We now apply the identity (2.81) to evaluate the integral on the right-most side of (3.277). We set $\chi=f^m_{\theta^{2p}t^c}$, $\omega=f^{m+1}_{\theta^{2p}t^c}$, $F=|g^m|^{2p}$, $\psi=R^m$, $g=g^m$, $G=G^{2p,c}$ and $h=h^{2p,c+1}$. Thus, using (3.277), we obtain

$$\begin{aligned}&\int_0^t\int_{\Gamma^m}\partial_{s^{2p}}\partial^*_{t^{c+1}}u^{m+1}\partial_{s^{2p}}\partial^*_{t^c}[u^{m+1}_n]^+_-=-\int_0^t\int_{\mathbb{S}^1}\Big\{|f^{m+1}_{\theta^{2p+1}t^{c+1}}|^2-|f^{m+1}_{\theta^{2p}t^{c+1}}|^2\Big\}-\epsilon\int_0^t\int_{\mathbb{S}^1}\Big\{|f^{m+1}_{\theta^{2p+3}t^{c+1}}|^2-|f^{m+1}_{\theta^{2p+2}t^{c+1}}|^2\Big\}\\&-\int_0^t\int_{\mathbb{S}^1}S(f^m_{\theta^{2p}t^c},f^{m+1}_{\theta^{2p}t^c},R^m,|g^m|^{2p}),\end{aligned}\tag{3.278}$$

where S is defined by (2.76). For the sake of simplicity, let us denote $\tilde S^m_{2p,c}:=S(f^m_{\theta^{2p}t^c},f^{m+1}_{\theta^{2p}t^c},R^m,|g^m|^{2p+1})$.

Estimating $\int_0^t\int_{\mathbb{S}^1}\tilde S^m_{2p,c}$. The first four terms in the expression for $\int_0^t\int_{\mathbb{S}^1}\tilde S^m_{2p,c}$ are cross-terms (given by the first four terms on RHS of (2.76)). We proceed analogously to the estimates (3.191) - (3.194):

$$\begin{aligned}&\Big|\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p}f^{m+1}_t\big(\partial_c^{2p}f^{m+1}_t-\partial_c^{2p}f^m_t\big)\Big|\le\lambda\int_0^t||\partial_c^{2p}f^m_t||^2_{L^2}+\frac{\tilde C}{\lambda}\int_0^t||\partial_c^{2p}f^{m+1}_t||^2_{L^2}\le\lambda\int_0^t||\partial_c^{2p}f^m_t||^2_{L^2}\\&+\frac{\tilde C\eta}{\lambda}\int_0^t||\nabla u^{m+1}_{t^c}||^2_{H^{2p-1}}+\frac{\tilde C}{\eta\lambda\epsilon}\int_0^t||\nabla u^{m+1}_{t^c}||^2_{H^{2p-2}}\le\lambda\int_0^t\mathfrak{D}^m_p+\frac{\tilde C\eta}{\lambda}\int_0^t\mathfrak{D}^{m+1}_p+\frac{\tilde Ct}{\eta\lambda\epsilon}\mathfrak{E}^{m+1}_p,\end{aligned}\tag{3.279}$$

where we used the inequality (3.253) in the last estimate. The second, third and the fourth term in the expression for $\int_0^t\int_{\mathbb{S}^1}\tilde S^m_{2p,c}$ are estimated in fully analogous manner. We obtain

$$\begin{aligned}&\Big|\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p}f^{m+1}_{\theta t}\big(\partial_c^{2p}f^{m+1}_{\theta t}-\partial_c^{2p}f^m_{\theta t}\big)\Big|+\epsilon\Big|\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p}f^{m+1}_{\theta^2t}\big(\partial_c^{2p}f^{m+1}_{\theta^2t}-\partial_c^{2p}f^m_{\theta^2t}\big)\Big|\\&+\epsilon\Big|\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p}f^{m+1}_{\theta^3t}\big(\partial_c^{2p}f^{m+1}_{\theta^3t}-\partial_c^{2p}f^m_{\theta^3t}\big)\Big|\le\lambda\int_0^t\mathfrak{D}^m_p+\frac{\tilde C\eta}{\lambda}\int_0^t\mathfrak{D}^{m+1}_p+\frac{\tilde Ct}{\eta\lambda\epsilon}\mathfrak{E}^{m+1}_p.\end{aligned}\tag{3.280}$$

As to the fifth term, we have:

$$\begin{aligned}&\Big|\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p}f^m_t\partial_c^{2p}f^{m+1}_t\big(\frac{R^m}{|g^m|^{4p}}-1\big)\Big|\le||\frac{R^m}{|g^m|^{4p}}-1||_{L^\infty}\int_0^t\Big(||\partial_c^{2p}f^m_t||^2_{L^2}+||\partial_c^{2p}f^{m+1}_t||^2_{L^2}\Big)\\&\le C||f^m||_{H^2}\int_0^t\mathfrak{D}^m_p+C||f^m||_{H^2}\int_0^t\mathfrak{D}^{m+1}_p,\end{aligned}\tag{3.281}$$

where we used the estimate analogous to (3.258) to bound $||\frac{R^m}{|g^m|^{4p}}-1||_{L^\infty}$. In a fully analogous manner we

estimate the sixth, seventh and the eighth term in the expression for $\int_0^t\int_{\mathbb{S}^1}\tilde{S}^m_{2p,c}$, where S is given by (2.76):

$$\begin{aligned}&\Big|\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p}f^m_{\theta t}\partial_c^{2p}f^{m+1}_{\theta t}\big(\frac{1}{|g^m|^{4p+1}}-1\big)\Big|+\epsilon\Big|\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p}f^m_{\theta^2 t}\partial_c^{2p}f^{m+1}_{\theta^2 t}\big(\frac{1}{|g^m|^{4p}}-1\big)\Big|\\&+\epsilon\Big|\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p}f^m_{\theta^3 t}\partial_c^{2p}f^{m+1}_{\theta^3 t}\big(\frac{1}{R^m|g^m|^{4p+1}}-1\big)\Big|\leq C||f^m||_{H^2}\int_0^t\mathfrak{D}^m_p+C||f^m||_{H^2}\int_0^t\mathfrak{D}^{m+1}_p.\end{aligned}\tag{3.282}$$

The last two terms in the expression for $\int_0^t\int_{\mathbb{S}^1}\tilde{S}^m_{2p,c}$ involve the quantities $G^{2p+1,c}$ and $h^{2p,c+1}$, defined through (3.246) and (3.249) respectively. Similarly to (3.275), the following inequality holds

$$\begin{aligned}&\Big|\int_0^t\int_{\mathbb{S}^1}\Big(\frac{f^m_{\theta^{2p}t^c}}{|g^m|^{2p}}+\frac{f^m_{\theta^{2p+2}t^c}}{R^m|g^m|^{2p+1}}\Big)h^{2p,c+1}\Big|+\Big|\int_0^t\int_{\mathbb{S}^1}\Big(\frac{f^{m+1}_{\theta^{2p}t^{c+1}}R^m}{|g^m|^{2p+1}}+\epsilon\frac{f^{m+1}_{\theta^{2p+4}t^{c+1}}}{|g^m|^{2p}}\Big)G^{2p,c+1}\Big|\\&+\Big|\int_0^t\int_{\mathbb{S}^1}h^{2p,c+1}G^{2p+1,c+1}\Big|\leq C(\sqrt{\mathfrak{E}^m}+||f^m||_{H^l})\big(\int_0^t\mathfrak{D}^m+\int_0^t\hat{\mathfrak{D}}^m\big)+C(\sqrt{\mathfrak{E}^m}+||f^m||_{H^l})\big(\int_0^t\mathfrak{D}^{m+1}+\int_0^t\hat{\mathfrak{D}}^{m+1}\big).\end{aligned}\tag{3.283}$$

3.2.6 Proof of Lemma 3.2

Combining the estimates (3.232), (3.238), (3.241), (3.254), (3.255), (3.256), (3.263) - (3.264), (3.279) - (3.283) together with the identity (3.203), we conclude for any $0\leq p\leq l-1$, $0\leq t\leq T\leq 1$

$$\begin{aligned}&\mathfrak{E}^{m+1}_{\epsilon,p}+\int_0^t\mathfrak{D}^{m+1}_{\epsilon,p}\leq\mathfrak{E}_{\epsilon,p}(0)+C\eta\int_0^t\mathfrak{D}^{m+1}_p+\frac{C\lambda}{\eta}\int_0^t\mathfrak{D}^{m+1}_p+C\big(\lambda+\sqrt{\mathfrak{E}^m}\big)\sum_{q=p+1}^{l-1}\int_0^t\mathfrak{D}^{m+1}_q+\frac{\tilde{C}t}{\lambda}\big[\mathfrak{E}^m_p+\mathfrak{E}^{m+1}_p\big]\\&+C(\sqrt{\mathfrak{E}^m}+\sup_{0\leq s\leq t}||f^m||_{H^4}+\lambda)\big[\int_0^t\mathfrak{D}^m+\int_0^t\hat{\mathfrak{D}}^m+\int_0^t\mathfrak{D}^{m+1}_p+\int_0^t\hat{\mathfrak{D}}^{m+1}_p\big]+Ct\int_0^t\hat{\mathfrak{D}}^{m+1}_p\\&+\frac{C}{\lambda}\sum_{q=0}^{p-1}\int_0^t\mathfrak{D}^{m+1}_q+C\sup_{0\leq s\leq t}||f^m||_{H^4}\big(\hat{\mathfrak{E}}^m_p+\hat{\mathfrak{E}}^{m+1}_p\big)+C(t+1)\sqrt{\mathfrak{E}^m}\big(\hat{\mathfrak{E}}^m_p+\hat{\mathfrak{E}}^{m+1}_p\big)\\&+\frac{\tilde{C}C_At\epsilon}{\lambda}\int_0^t\mathfrak{D}^m+\frac{\tilde{C}C_At^2\epsilon}{\lambda}\mathfrak{D}^{m+1}+\frac{\tilde{C}C_At\epsilon}{\lambda}\theta_1+\frac{\tilde{C}t}{\lambda\eta\epsilon}\mathfrak{E}^{m+1}_p+\tilde{C}t\sup_{0\leq s\leq t}||f^m||_{H^2}\big[\mathfrak{E}^m_p+\mathfrak{E}^{m+1}_p\big]\\&+\tilde{C}\sup_{0\leq s\leq t}||f^m||_{H^2}\big(\frac{t}{\epsilon^2}\mathfrak{E}^{m+1}_p+\int_0^t\mathfrak{D}^{m+1}_p\big)+\frac{C}{\eta\lambda}\int_0^t\mathcal{D}^{m+1}+\frac{\tilde{C}\lambda t}{\epsilon^4}\sup_{0\leq s\leq t}\mathcal{E}^{m+1}+Ct\epsilon\mathcal{E}^m_{(0)}+\frac{\tilde{C}C_At}{\lambda}\theta_1\\&+C\epsilon\sup_{0\leq s\leq t}||f^m||^2_{H^4}\mathcal{E}^m+C\epsilon||\mathbb{P}_1f^m||^4_{L^2},\end{aligned}\tag{3.284}$$

where we employ the convention $\sum_a^b\cdots=0$ if $a,b\in\mathbb{N}$ and $a>b$ and $\mathfrak{D}^{m+1}_l:=0$. We choose λ, η small, and allow the constants C to become larger (depending on the choice of λ and η). Further, recall the smallness assumption $\sup_{0\leq s\leq t}\big(\mathcal{E}(s)+\mathfrak{E}(s)\big)+\int_0^t\{\mathcal{D}(s)+\mathfrak{D}(s)\}\,ds\leq\theta_2$ introduced in the paragraph after (3.197). We conclude that there exist constants $\tilde{C}_1$, $\tilde{C}_2$ and C_2 such that for any $0\leq p\leq l-1$:

$$\begin{aligned}&\big(1-\tilde{C}_1t-\frac{\tilde{C}_1t}{\epsilon}\big)\big(\mathfrak{E}^{m+1}_p+\int_0^t\mathfrak{D}^{m+1}_p\big)+\big(1-\tilde{C}_2t-C_2t\big)\big(\hat{\mathfrak{E}}^{m+1}_p+\int_0^t\hat{\mathfrak{D}}^{m+1}_p\big)\\&\quad\leq\mathfrak{E}_{\epsilon,p}(0)+\nu^l\sum_{q=p+1}^{l-1}\int_0^t\mathfrak{D}^{m+1}_q+C\sum_{q=0}^{p-1}\int_0^t\mathfrak{D}^{m+1}_q+\mathcal{J}^p+\frac{\tilde{C}\lambda t}{\epsilon^4}\sup_{0\leq s\leq t}\mathcal{E}^{m+1}+C^*\int_0^t\mathcal{D}^{m+1},\end{aligned}\tag{3.285}$$

whereby $\nu>0$ is a small, but fixed number and

$$\begin{aligned}\mathcal{J}^p:=&\frac{\tilde{C}t}{\lambda}\mathfrak{E}^m_p+C\sqrt{\mathfrak{E}^m}\hat{\mathfrak{E}}^m_p+\frac{\tilde{C}C_At\epsilon}{\lambda}\int_0^t\mathfrak{D}^m+\tilde{C}t\sup_{0\leq s\leq t}||f^m||_{H^2}\mathfrak{E}^m_p\\&+\frac{\tilde{C}C_At\epsilon}{\lambda}\theta_1+Ct\epsilon\mathcal{E}^m_{(0)}+C\epsilon\sup_{0\leq s\leq t}||f^m||^2_{H^4}\mathcal{E}^m+C\epsilon||\mathbb{P}_1f^m||^4_{L^2}.\end{aligned}$$

Furthermore, let $D_1 := 1-\tilde{C}_1 t - \frac{\tilde{C}_1 t}{\epsilon}$, $D_2 := 1-\tilde{C}_2 t - C_2 t$. For each $0\le p\le l-1$, multiply (3.285) by ν^p and sum over all $p=0,1,\dots,l-1$. We obtain

$$D_1\sum_{p=0}^{l-1}\nu^p\mathfrak{E}_p^{m+1} + D_1\sum_{p=0}^{l-1}\nu^p\int_0^t\mathfrak{D}_p^{m+1} + D_2\sum_{p=0}^{l-1}\nu^p\big(\hat{\mathfrak{E}}_p^{m+1}+\int_0^t\hat{\mathfrak{D}}_p^{m+1}\big) \le \sum_{p=0}^{l-1}\nu^p\mathfrak{E}_{\epsilon,p}(0) + \sum_{p=0}^{l-1}\nu^p\mathcal{J}^p$$
$$+\nu^l\sum_{p=0}^{l-1}\nu^p\sum_{q=p+1}^{l-1}\int_0^t\mathfrak{D}_q^{m+1} + C\sum_{p=0}^{l-1}\nu^p\sum_{q=0}^{p-1}\int_0^t\mathfrak{D}_q^{m+1} + \sum_{p=0}^{l-1}\nu^p\Big(\frac{\tilde{C}\lambda t}{\epsilon^4}\sup_{0\le s\le t}\mathcal{E}^{m+1} + C^*\int_0^t\mathcal{D}^{m+1}\Big).$$

From here we infer

$$D_1\sum_{p=0}^{l-1}\nu^p\mathfrak{E}_p^{m+1} + \big(D_1-\sum_{q=1}^{l-1}\nu^q\big)\int_0^t\mathfrak{D}_0^{m+1} + \sum_{p=1}^{l-1}\big(D_1\nu^p-\nu^l\sum_{q=0}^{p-1}\nu^q-\sum_{q=p+1}^{l-1}\nu^q\big)\int_0^t\mathfrak{D}_p^{m+1}$$
$$+D_2\sum_{p=0}^{l-1}\nu^p\big(\hat{\mathfrak{E}}_p^{m+1}+\int_0^t\hat{\mathfrak{D}}_p^{m+1}\big) \le \sum_{p=0}^{l-1}\nu^p\mathfrak{E}_{\epsilon,p}(0) + \sum_{p=0}^{l-1}\nu^p\mathcal{J}^p + \sum_{p=0}^{l-1}\nu^p\Big(\frac{\tilde{C}\lambda t}{\epsilon^4}\sup_{0\le s\le t}\mathcal{E}^{m+1} + C^*\int_0^t\mathcal{D}^{m+1}\Big)$$

Let us set $d_0 = D_1 - \sum_{q=1}^{l-1}\nu^q$ and for $1\le p\le l-1$ set $d_p = D_1\nu^p - \nu^l\sum_{q=0}^{p-1}\nu^q - \sum_{q=p+1}^{l-1}\nu^q$. From the previous inequality we immediately obtain

$$D_1\mathfrak{E}^{m+1,\nu} + \sum_{p=0}^{l-1}d_p\int_0^t\mathfrak{D}_p^{m+1} + D_2\hat{\mathfrak{E}}^{m+1,\nu} + D_2\int_0^t\hat{\mathfrak{D}}^{m+1,\nu} \le \delta_0^\nu + \mathcal{J}^\nu + \sum_{p=0}^{l-1}\nu^p\Big(\frac{\tilde{C}\lambda t}{\epsilon^4}\sup_{0\le s\le t}\mathcal{E}^{m+1} + C^*\int_0^t\mathcal{D}^{m+1}\Big), \tag{3.286}$$

where $\delta_0^\nu := \sum_{p=0}^{l-1}\nu^p\mathfrak{E}_{\epsilon,p}(0)$ and $\mathcal{J}^\nu := \sum_{p=0}^{l-1}\nu^p\mathcal{J}^p$. Let $0<\alpha\le 1$ be a real number to be specified later. Multiplying the inequality (3.286) by α and adding it to (3.199), we arrive at

$$\begin{aligned}&\mathcal{E}_\epsilon^{m+1} + \int_0^t\mathcal{D}_\epsilon^{m+1} + \alpha\Big(D_1\sup_{0\le s\le t}\mathfrak{E}^{m+1,\nu} + \sum_{p=0}^{l-1}d_p\int_0^t\mathfrak{D}_p^{m+1}\Big) + \alpha D_2\big(\sup_{0\le s\le t}\hat{\mathfrak{E}}^{m+1,\nu} + \int_0^t\hat{\mathfrak{D}}^{m+1,\nu}\big)\\
&\le \mathcal{E}_\epsilon(0) + \mathcal{K}(\theta_2,t) + \frac{C_0 t}{\lambda}\big(\mathcal{E}^{m+1} + (\mathfrak{E}^{m+1})^2\big) + \big(\frac{\bar{C}t}{\eta\lambda} + \frac{\bar{C}t}{\epsilon\eta\lambda}\big)\mathcal{E}^{m+1} + \frac{\bar{C}t}{\lambda}\mathcal{E}_\epsilon^{m+1} + C\big(\lambda\theta_2 + \sqrt{\theta_2}\big)\int_0^t\mathcal{D}_\epsilon^{m+1}\\
&+\big(C\lambda + \frac{\bar{C}\eta}{\lambda} + \frac{\bar{C}\eta}{\epsilon\lambda} + C\theta_2^{1/2}\big)\int_0^t\mathfrak{D}^{m+1} + C\theta_2\sup_{0\le s\le t}\mathfrak{E}^{m+1} + \alpha\Big(\delta_0^\nu + \mathcal{J}^\nu + \sum_{p=0}^{l-1}\nu^p\Big(\frac{\tilde{C}\lambda t}{\epsilon^4}\sup_{0\le s\le t}\mathcal{E}^{m+1} + C^*\int_0^t\mathcal{D}^{m+1}\Big)\Big)\end{aligned} \tag{3.287}$$

Absorbing the factors of $\mathcal{E}_\epsilon^{m+1}$, $\int_0^t\mathcal{D}_\epsilon^{m+1}$ and $\mathfrak{E}_\epsilon^{m+1}+\int_0^t\mathfrak{D}_\epsilon^{m+1}$ from RHS into LHS and recalling that $\mathfrak{E}^\nu + \int_0^t\mathfrak{D}^\nu \le \mathfrak{E} + \int_0^t\mathfrak{D} \le \nu^{-(l-1)}\big(\mathfrak{E}^\nu + \int_0^t\mathfrak{D}^\nu\big)$, we obtain

$$\begin{aligned}&\big(1-\frac{C_0 t}{\lambda} - \frac{\bar{C}t}{\eta\lambda} - \frac{\bar{C}t}{\epsilon\eta\lambda} - \frac{\tilde{C}\lambda t}{\epsilon^4}\sum_{p=0}^{l-1}\nu^p\big)\mathcal{E}_\epsilon^{m+1} + \big(1 - C(\lambda+\theta_2+\sqrt{\theta_2}) - \alpha C^*\sum_{p=0}^{l-1}\nu^p\big)\int_0^t\mathcal{D}_\epsilon^{m+1}\\
&+(\alpha D_1 - C\theta_2\nu^{-(l-1)})\sup_{0\le s\le t}\mathfrak{E}^{m+1,\nu} + \sum_{p=0}^{l-1}\Big(\alpha d_p - C\lambda - \frac{\bar{C}\eta}{\lambda} - \frac{\bar{C}\eta}{\epsilon\lambda} - C\theta_2^{1/2} - \frac{C_0 t}{\lambda}\Big)\int_0^t\mathfrak{D}_p^{m+1}\\
&+\alpha D_2\big(\sup_{0\le s\le t}\hat{\mathfrak{E}}^{m+1,\nu} + \int_0^t\hat{\mathfrak{D}}^{m+1,\nu}\big) \le \mathcal{E}_\epsilon(0) + \mathcal{K}(\theta_2,t) + \alpha\delta_0^\nu + \alpha\mathcal{J}^\nu + \frac{C_0 t}{\lambda}\nu^{-2(l-1)}(\mathfrak{E}^{m+1,\nu})^2\end{aligned} \tag{3.288}$$

Now set $\alpha = \min(\frac{\theta_1}{\theta_2}, \frac{1}{20C^*\sum_{p=0}^{l-1}\nu^p}, 1)$ and choose λ, μ, θ_1, θ_2, ν and then $t = t^\epsilon$ so that

$$\frac{C_0 t}{\lambda} + \frac{\bar{C}t}{\eta\lambda} + \frac{\bar{C}t}{\epsilon\eta\lambda} + \frac{\tilde{C}\lambda t}{\epsilon^4}\sum_{p=0}^{l-1}\nu^p < 1/10;\ C(\lambda+\theta_2+\sqrt{\theta_2}) + \frac{1}{20} < \frac{1}{10};\ \alpha D_1 - C\theta_2\nu^{-(l-1)} \ge \frac{9}{10}\alpha;$$
$$\alpha d_p - C\lambda - \frac{\bar{C}\eta}{\lambda} - \frac{\bar{C}\eta}{\epsilon\lambda} - C\theta_2^{1/2} - \frac{C_0 t}{\lambda} > \frac{9}{10}\alpha\nu^p;\ \mathcal{K}(\theta_2,t) < \frac{\theta_1}{10};\ \alpha\mathcal{J}^\nu < \frac{\theta_1}{10};\ \frac{C_0 t}{\lambda}\nu^{-2(l-1)} < \frac{1}{10}\alpha^2.$$

Let $\theta_0=\theta_2\nu^{l-1}$ and assume $\mathcal{E}_\epsilon(0)+\alpha\delta_0^\nu\le\theta_0/2$ (and hence $\mathcal{E}_\epsilon(0)+\alpha\delta_0^\nu\le\theta_2/2$), we obtain from (3.288)

$$\sup_{0\le s\le t}\mathcal{E}_\epsilon^{m+1}+\int_0^t\mathcal{D}_\epsilon^{m+1}+\alpha\big(\sup_{0\le s\le t}\mathfrak{E}_\epsilon^{m+1,\nu}+\int_0^t\mathfrak{D}_\epsilon^{m+1,\nu}\big)\le\frac{7}{9}\theta_2+\frac{1}{9}\big(\alpha\sup_{0\le s\le t}(\mathfrak{E}_\epsilon^{m+1,\nu}\big)^2.$$

From the previous inequality we conclude that there exists a (possibly smaller) t^ϵ such that the first claim of (3.115) holds. This finishes the proof of Lemma 3.2.

3.3 Regularized Stefan problem

The principal goal of this subsection is the following local existence theorem:

Theorem 3.4 *Let $\zeta>0$. For any sufficiently small $L>0$ there exists $t^\epsilon>0$ depending on L and ϵ and constants $0<\alpha,\nu\le1$ such that if*

$$E_{\alpha,\nu}(u_0^\epsilon,f_0^\epsilon)\le\frac{L}{2}$$

then there exists a unique solution (u^ϵ,f^ϵ) to the regularized Stefan problem defined on the time interval $[0,t^\epsilon[$, satisfying the bound

$$\sup_{0\le t\le t^\epsilon}E_{\alpha,\nu}(u^\epsilon,f^\epsilon)(t)+\int_0^{t^\epsilon}D_{\alpha,\nu}(u^\epsilon,f^\epsilon)(\tau)\,d\tau\le L$$

and $E_{\alpha,\nu}(u^\epsilon,f^\epsilon)(\cdot)$ is continuous on $[0,t^\epsilon[$.

3.3.1 Local existence for the regularized Stefan problem on $[0,t^\epsilon[$

Having obtained the uniform bounds on the sequence (u^m,R^m), our goal in this section is to prove that this sequence is actually Cauchy in the energy space. In order to do that, we need to compare two consecutive iterates in the energy norm. However, the expression $\mathcal{E}_\epsilon(u^{m+1}-u^m,R^{m+1}-R^m)$, does not a-priori make sense, because the functions $u_\pm^{m+1}$ and $u_\pm^m$ are defined on *different* domains Ω^m and Ω^{m-1}, respectively. To overcome this difficulty, we shall pull-back the functions u^k from Ω^{k-1} onto $\Omega_{S_1(x_0,0)}$, where we recall $\Omega_{S_1(x_0,0)}=\Omega\setminus S_1(x_0,0)$. To apply this change of variables, we first define the map $\bar{\mathbf{x}}=\Theta^m(t,\mathbf{x})$, where $\Theta^m(t,\mathbf{x})=(x_0,0)+\pi^m(t,\mathbf{x})(\mathbf{x}-(x_0,0))$ and π^m is a smooth scalar-valued function with the property

$$\pi^m(t,\mathbf{x})=\begin{cases}1/R^{m-1}(\frac{\mathbf{x}-(x_0,0)}{|\mathbf{x}-(x_0,0)|}), & |\mathbf{x}-(x_0,0)|-1\le d,\\ 1, & |\mathbf{x}-(x_0,0)|-1\ge 2d\end{cases}\quad d\quad\text{is small.}\tag{3.289}$$

By abuse of notation, we shall interchangeably use the fact that $R^{m-1}(\frac{\mathbf{x}-(x_0,0)}{|\mathbf{x}-(x_0,0)|})=R^{m-1}(\theta)$, for a unique θ such that $(\cos\theta,\sin\theta)=\frac{\mathbf{x}-(x_0,0)}{|\mathbf{x}-(x_0,0)|}$. For any $\mathbf{x}\in\Gamma^{m-1}$, we have $\mathbf{x}=(x_0,0)+R^{m-1}(\theta)\frac{\mathbf{x}-(x_0,0)}{|\mathbf{x}-(x_0,0)|}$, by the definition of Γ^{m-1}. Hence

$$\begin{aligned}\Theta^m(t,\mathbf{x}) &= (x_0,0)+\pi^m(t,R^{m-1}(\theta)\frac{\mathbf{x}-(x_0,0)}{|\mathbf{x}-(x_0,0)|})R^{m-1}(\theta)\frac{\mathbf{x}-(x_0,0)}{|\mathbf{x}-(x_0,0)|}=(x_0,0)+\frac{1}{R^{m-1}(\theta)}R^{m-1}(\theta)\frac{\mathbf{x}-(x_0,0)}{|\mathbf{x}-(x_0,0)|}\\ &= (x_0,0)+\frac{\mathbf{x}-(x_0,0)}{|\mathbf{x}-(x_0,0)|}\in S_1(x_0,0).\end{aligned}$$

Thus Θ^m does the job for us: $\Theta^m:\Omega^{m-1}\to\Omega_{S_1(x_0,0)}$. Note that this map is natural from the point of view of our *geometric* assumption that the evolving interfaces Γ^{m-1} are, in fact, 'graphs' over the unit sphere $S_1(x_0,0)$ centered at $(x_0,0)$. For notational simplicity, from now on we shall only work with the case $x_0=0$, i.e. $S_1(x_0,0)=\mathbb{S}^1$ and we shall also drop the boldface notation and denote the vectors $\mathbf{x}$ and $\bar{\mathbf{x}}$ by x and $\bar{x}$ respectively.

The inverse map $x(\bar{x})$ to the above change of variables (3.289) is given by

$$x=\rho^m(\bar{x})\bar{x}.$$

To convince ourselves that the function $\rho^m:\Omega_{\mathbb{S}^1}\to\mathbb{R}$ is well defined, we observe that ρ^m has to satisfy the relation $\pi^m(t,\rho^m(t,\bar{x})\bar{x})\rho^m(\bar{x})-1=0$. The existence of such a ρ^m can be established by the implicit function theorem, applied to the equation $F(t,s,\bar{x})=0$, where $F:\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ is given by $F(t,s,v)=\pi^m(t,sv)s-1$. Namely, $F_s(t,s,v)=(\nabla\pi^m\cdot v)s+\pi^m>0$ since $\pi^m\approx 1$ and $\nabla\pi^m=O(f_\theta^{m-1})$ is small. We define $\bar{u}^m:\Omega_{\mathbb{S}^1}\to\mathbb{R}$ by setting

$$\bar{u}^m(\bar{x}):=u^m(t,\rho^m(t,\bar{x})\bar{x}). \tag{3.290}$$

The heat operator $\partial_t-\Delta$ on the domain Ω^{m-1} will transform into a more complicated operator in the new coordinates. The following lemma describes this description in the new coordinates:

Lemma 3.5 *The push forward of the operator $\partial_t-\Delta$ with respect to the map $\Theta^m:\Omega^{m-1}\to\mathbb{S}^1$ reads:*

$$(\partial_t-\Delta)^{\#}\bar{u}=\bar{u}_t-\Delta_{\bar{x}}\bar{u}-a_{ij}^m\bar{u}_{\bar{x}^i\bar{x}^i}-b_i^m\bar{u}_{\bar{x}^i},$$

where

$$a_{ij}^m:=((\pi^m)^2-1)\delta_{ij}+2\pi^m x^j\pi_{x^i}^m+x^ix^j|\nabla\pi^m|^2,\quad b_i^m:=2\pi_{x^i}^m+\Delta\pi^m x^i-\pi_t^m x^i. \tag{3.291}$$

Furthermore,

$$[\bar{u}_n^{m+1}]_-^+=\frac{(R^m)^2}{|g^m|}\Big([u_n^{m+1}]_-^+\Big)\circ\phi^m \tag{3.292}$$

Proof. Note that (3.289) implies (e_i is the i-th unit vector, $i=1,2$)

$$\bar{x}_{x^i}=\pi^m e_i+\pi_{x^i}^m x. \tag{3.293}$$

From $\pi^m(x)=1/\rho^m(\bar{x})$, we obtain $\pi_{x^i}^m=-(\nabla\rho^m\cdot\bar{x}_{x^i})/(\rho^m)^2$, which in turn, combined with (3.293), after an elementary calculation implies

$$\pi_{x^i}^m=-\frac{\rho_{\bar{x}^i}^m}{(\rho^m)^2\Psi^m}, \tag{3.294}$$

where $\Psi^m=\rho^m-\nabla\rho^m\cdot\bar{x}$. After further differentiating (3.294) with repspect to x^i and using the relation (3.293), we arrive at

$$\Delta\pi^m=\frac{-\Delta\rho^m}{(\rho^m)^3\Psi^m}+\frac{2|\nabla\rho^m|^2\Psi^m-2|\nabla\rho^m|^2\nabla\rho^m\cdot\bar{x}+\rho^m|\nabla\rho^m|^2(\Psi^m)^{-1}\bar{x}^j\bar{x}^k\rho_{\bar{x}^j\bar{x}^k}^m}{(\rho^m)^4(\Psi^m)^2}.$$

Similarly, it is straightforward to see that $\pi_t^m=-\rho_t^m/(\rho^m)^2$. In order to evaluate the Laplacian in new coordinates, by (3.290) we first write

$$u^m(t,x)=\bar{u}^m(t,\pi^m(t,x)x). \tag{3.295}$$

Applying Δ_x to the left-hand side, we obtain

$$\Delta u^m(t,x)=(\pi^m)^2\Delta_{\bar{x}}\bar{u}^m+\big(2\pi^m x^j\pi_{x^i}^m+x^ix^j|\nabla\pi^m|^2\big)\bar{u}_{\bar{x}^i\bar{x}^j}^m+\big(2\pi_{x^i}^m+\Delta\pi^m x^i\big)\bar{u}_{x^i}^m. \tag{3.296}$$

Applying ∂_t to the left hand side of (3.295), we obtain

$$u_t^m(t,x)=\bar{u}_t^m+\pi_t^m\nabla_{\bar{x}}\bar{u}^m\cdot x=\bar{u}_t^m-\frac{\rho_t^m}{(\rho^m)^2}\nabla_{\bar{x}}\bar{u}^m\cdot\bar{x}=\bar{u}_t^m-\frac{\rho_t^m}{\rho^m}\bar{x}_i\bar{u}_{\bar{x}^i}^m. \tag{3.297}$$

Since $(\partial_t-\Delta)u^m=0$, we conclude from (3.296) and (3.297)

$$(\partial_t-\Delta_{\bar{x}})\bar{u}^m=a_{ij}^m\bar{u}_{\bar{x}^i\bar{x}^i}^m+b_i^m\bar{u}_{\bar{x}^i}^m,$$

where a_{ij}^m and b_i^m are given by (3.291). Next, using (3.294), it is easy to see that

$$a_{ij}^m=\frac{1-(\rho^m)^2}{(\rho^m)^2}\delta_{ij}-2\frac{\rho_{\bar{x}^i}^m\bar{x}^j}{(\rho^m)^2\Psi^m}+\frac{\bar{x}^i\bar{x}^j|\nabla\rho^m|^2}{(\rho^m)^2(\Psi^m)^2}.$$

We now turn to the proof of (3.292). Note that

$$u^{m+1}_{x^i} = \nabla_{\bar{x}} \bar{u}^{m+1} \bar{x}_{x^i} = \nabla_{\bar{x}} \bar{u}^{m+1} \big(\pi^{m+1} e_i + \pi^{m+1}_{x^i} x\big) = \nabla_{\bar{x}} \bar{u}^{m+1} \big(\frac{e_i}{\rho^{m+1}} - \frac{\rho^{m+1}_{\bar{x}^i}}{(\rho^{m+1})^2 \Psi^{m+1}} \rho^{m+1} \bar{x}\big).$$

From here, we infer the formula

$$\nabla u^{m+1}(x) = \frac{1}{\rho^{m+1}} \nabla \bar{u}^{m+1}(\bar{x}) - \frac{\nabla \rho^{m+1}(\bar{x})}{\rho^{m+1}(\bar{x}) \Psi^{m+1}(\bar{x})} \bar{x} \cdot \nabla \bar{u}^{m+1}(\bar{x}).$$

Denote $\xi = (\cos\theta, \sin\theta)$ and let n and τ stand for the unit normal and unit tangent on $\mathbb{S}^1$ respectively. From the above formula, we obtain

$$[\nabla u^{m+1}]^+_- \circ \phi^m(\theta) = \frac{1}{\rho^{m+1}(\xi)} [\nabla \bar{u}^{m+1}]^+_-(\xi) - \frac{\rho^{m+1}_\theta \tau}{\rho^{m+1}(\xi) \Psi^{m+1}(\xi)} \xi \cdot [\nabla \bar{u}^{m+1}]^+_-(\xi) \tag{3.298}$$

Note that

$$\xi \cdot [\nabla \bar{u}^{m+1}]^+_-(\xi) = \xi \cdot \big(\partial_\theta [\bar{u}^{m+1}]^+_- \tau + [\bar{u}^{m+1}_n]^+_- n\big) = [\bar{u}^{m+1}_n]^+_-,$$

since $\partial_\theta [\bar{u}^{m+1}]^+_- = 0$ (due to the fact that $u^{m+1,+}|_{\mathbb{S}^1} = u^{m+1,-}|_{\mathbb{S}^1}$) and $\xi \cdot n = |n|^2 = 1$. Observe further that $\rho^{m+1}(\xi) = R^m(\theta)$. Furthermore, since $\nabla \rho^{m+1}(\xi) \cdot \xi = \partial_\theta \rho^{m+1} \tau \cdot \xi = 0$, we also have $\Psi^{m+1}(\xi) = \rho^{m+1}(\xi) = R^m(\theta)$. Form these observations and from (3.298) we obtain the formula

$$[\nabla u^{m+1}]^+_- \circ \phi^m = \frac{1}{\rho^{m+1}} [\bar{u}^{m+1}_n]^+_- n - \frac{\rho^{m+1}_\theta \tau}{\rho^{m+1} \Psi^{m+1}} [\bar{u}^{m+1}_n]^+_- = \frac{1}{R^m} [\bar{u}^{m+1}_n]^+_- n - \frac{f^m_\theta}{(R^m)^2} [\bar{u}^{m+1}_n]^+_- \tau.$$

It is straightforward to see that $n_{\Gamma^m} \circ \phi^m = \frac{R^m}{|g^m|} n - \frac{f^m_\theta}{|g^m|} \tau$, where τ and n stand for the unit tangent and the unit normal on $\mathbb{S}^1$, respectively. Hence

$$\begin{aligned}
&\Big([\nabla u^{m+1}]^+_- \cdot n_{\Gamma^m}\Big) \circ \phi^m = \Big(\frac{1}{R^m} [\bar{u}^{m+1}_n]^+_- n - \frac{f^m_\theta}{(R^m)^2} [\bar{u}^{m+1}_n]^+_- \tau\Big) \cdot \Big(\frac{R^m}{|g^m|} n - \frac{f^m_\theta}{|g^m|} \tau\Big) \\
&= \frac{1}{|g^m|} [\bar{u}^{m+1}_n]^+_- + \frac{(f^m_\theta)^2}{|g^m| (R^m)^2} [\bar{u}^{m+1}_n]^+_- = \frac{|g^m|}{(R^m)^2} [\bar{u}^{m+1}_n]^+_-
\end{aligned}$$

and this proves (3.292). □

Since $\Big([\nabla u^{m+1}]^+_- \cdot n_{\Gamma^m}\Big) \circ \phi^m = \frac{-f^{m+1}_t R^m}{|g^m|} + \epsilon \frac{-f^{m+1}_{\theta^4 t}}{|g^m|}$, from (3.292), we immediately obtain

$$\begin{aligned}
[\bar{u}^{m+1}_n]^+_- &= -\frac{f^{m+1}_t (R^m)^3}{|g^m|^2} - \epsilon \frac{f^{m+1}_{\theta^4 t} (R^m)^2}{|g^m|^2} = -\frac{f^{m+1}_t R^m}{|g^m|} - \epsilon \frac{f^{m+1}_{\theta^4 t}}{|g^m|} + \Big(\frac{f^{m+1}_t R^m}{|g^m|} - \epsilon \frac{f^{m+1}_{\theta^4 t}}{|g^m|}\Big)\big(\frac{(R^m)^2}{|g^m|} - 1\big) \\
&= -\frac{f^{m+1}_t R^m}{|g^m|} - \epsilon \frac{f^{m+1}_{\theta^4 t}}{|g^m|} - \Xi(f^m),
\end{aligned} \tag{3.299}$$

where

$$\Xi(f^m) \equiv \Xi^m := \Big(\frac{f^{m+1}_t R^m}{|g^m|} + \epsilon \frac{f^{m+1}_{\theta^4 t}}{|g^m|}\Big)\big(\frac{(R^m)^2}{|g^m|} - 1\big) = \big(1 - \frac{|g^m|}{(R^m)^2}\big) [\bar{u}^{m+1}_n]^+_-.$$

For any $k \in \mathbb{N}$, let $v^{k+1} := \bar{u}^{k+1} - \bar{u}^k$ and $\sigma^{k+1} = f^{k+1} - f^k$. By subtracting two consecutive equations in the iteration process, we obtain:

$$v^{m+1}_t - \Delta v^{m+1} = f^\circ_m \quad \text{on} \quad \Omega^m \tag{3.300}$$

$$v^{m+1} = -\sigma^m - \frac{\sigma^m_{\theta^2}}{R^m |g^m|} + G^\circ_m \quad \text{on} \quad \mathbb{S}^1, \tag{3.301}$$

$$v^{m+1}_n = 0 \quad \text{on} \quad \partial\Omega, \tag{3.302}$$

$$[v^{m+1}_n]^+_- = -\frac{\sigma^{m+1}_t R^m}{|g^m|} - \epsilon \frac{\sigma^{m+1}_{\theta^4 t}}{|g^m|} + h^\circ_m \quad \text{on} \quad \mathbb{S}^1. \tag{3.303}$$

Here

$$f_m^\circ = a_{ij}^{m+1}\bar{u}_{ij}^{m+1} + b_i^{m+1}\bar{u}_i^{m+1} - a_{ij}^m\bar{u}_{ij}^m + b_i^m\bar{u}_i^m,$$
$$G_m^\circ = N^*(f^m) - N^*(f^{m-1}),$$
$$h_m^\circ = -f_t^m\Big(\frac{R^m}{|g^m|} - \frac{R^{m-1}}{|g^{m-1}|}\Big) - \epsilon f_{\theta^4 t}^m\Big(\frac{1}{|g^m|} - \frac{1}{|g^{m-1}|}\Big) - \Xi^m + \Xi^{m-1}.$$

Here, the quadratic nonlinearity $N^*(\cdot)$ is defined in the line after (3.96). Upon applying the differential operator ∂_{t^k} to the equations (3.300), (3.301) and (3.303) and singling out the leading order terms, we arrive at:

$$v_{t^{k+1}}^{m+1} - \Delta v_{t^k}^{m+1} = f_m', \tag{3.304}$$

$$v_{t^k}^{m+1} = -\sigma_{t^k}^m - \frac{\sigma_{\theta^2 t^k}^m}{R^m|g^m|} + G_m' \quad \text{on} \quad \mathbb{S}^1, \tag{3.305}$$

$$[\partial_n v_{t^k}^{m+1}]_-^+ = -\sigma_{t^{k+1}}^{m+1}\frac{R^m}{|g^m|} - \epsilon\frac{\sigma_{\theta^4 t^{k+1}}^{m+1}}{|g^m|} + h_m' \quad \text{on} \quad \mathbb{S}^1, \tag{3.306}$$

where

$$f_m' = \partial_{t^k} f_m^\circ \tag{3.307}$$

$$G_m' = -\sum_{q=0}^{k-1}\sigma_{\theta^2 t^q}^m\Big(\frac{1}{R^m|g^m|}\Big)_{t^{k-q}} + \partial_{t^k}G_m^\circ. \tag{3.308}$$

$$h_m' = -\sum_{q=0}^{k-1}\sigma_{t^{q+1}}^{m+1}\Big(\frac{R^m}{|g^m|}\Big)_{t^{k-q}} - \epsilon\sum_{q=0}^{k-1}\sigma_{\theta^4 t^{q+1}}^{m+1}\Big(\frac{1}{|g^m|}\Big)_{t^{k-q}} + \partial_{t^k}h_m^\circ. \tag{3.309}$$

In order to obtain the energy identities for the problem (3.304) - (3.306), we may use the identities from Section 2. Respecting the notations from (2.62) - (2.66) from Section 2, for any $j \in \mathbb{N}$ we set

$$\mathcal{F} = f_j', F \equiv 1, G = g_j', h = h_j', \mathcal{U} = v_{t^k}^{j+1}, \omega = \sigma_{t^k}^{j+1}, \chi = \sigma_{t^k}^j, \text{ and } \psi = R^j. \tag{3.310}$$

Note that $\Gamma = \mathbb{S}^1$ and hence, since the domain is not time-dependent, we have $V_\Gamma = 0$ and the terms $\tilde{\mathcal{U}}^\pm$ and $\tilde{\mathcal{V}}$ in equations (2.63) and (2.65) are simply equal to 0. In other words, the expressions $O(\mathcal{U})$ and $P(\mathcal{U})$ defined in (2.73) and (2.74) are equal to 0. We define the energy:

$$\bar{e}^j := \sum_{k=0}^{l-1}\bar{\mathcal{E}}_\epsilon(v^j,\sigma^j), \quad \bar{d}^j := \sum_{k=0}^{l-1}\bar{\mathcal{D}}_\epsilon(v^j,\sigma^j),$$

where $\bar{\mathcal{E}}_\epsilon$ and $\bar{\mathcal{D}}_\epsilon$ are defined by (2.67) and (2.68) respectively. Furthermore, using (2.71) and (3.310) and keeping in mind that $O = P = 0$, we arrive at

$$\bar{e}^{m+1}(t) + \int_0^t \bar{d}^{m+1}(s)\,ds = \int_0^t\int_{\mathbb{S}^1}\{q^m + s^m\}, \tag{3.311}$$

where $q^m = \sum_{k=0}^{l-1}Q(\sigma_{t^k}^m, \sigma_{t^k}^{m+1}, R^m)$, $s^m = \sum_{k=0}^{l-1}S(\sigma_{t^k}^m, \sigma_{t^k}^{m+1}, R^m)$. Here, Q and S are defined by (2.75) and (2.76) respectively. Just like in Section 3.2, for any $0 < \nu \le 1$, we introduce the higher order energy:

$$\mathfrak{e}^{j,\nu} := \mathfrak{E}^\nu(v^j,\sigma^j),\ \hat{\mathfrak{e}}^{j,\nu} := \hat{\mathfrak{E}}^{\epsilon,\nu}(v^j,\sigma^j), \quad \mathfrak{d}^{j,\nu} := \mathfrak{D}^\nu(v^j,\sigma^j),\ \hat{\mathfrak{d}}^{j,\nu} := \hat{\mathfrak{D}}^{\epsilon,\nu}(v^j,\sigma^j); \tag{3.312}$$

here $j \in \mathbb{N}$, $0 \le k \le l-1$. Like in the previous sections, we drop the index ν if $\nu = 1$. Before we explain the energy estimates, we first examine the form of f_m°, G_m° and h_m° featuring RHS of equations (3.300), (3.301) and (3.303) respectively. Note that f_m° can be rewritten in the form

$$f_m^\circ = (a_{ij}^{m+1} - a_{ij}^m)\bar{u}_{ij}^{m+1} + a_{ij}^m(\bar{u}_{ij}^{m+1} - \bar{u}_{ij}^m) + (b_i^{m+1} - b_i^m)\bar{u}_i^{m+1} + b_i^m(\bar{u}_i^{m+1} - \bar{u}_i^m).$$

We observe that $a_{ij}^{m+1} - a_{ij}^m$ can be written as $A_{ij}(\sigma^m, \sigma_\theta^m)$ for some smooth function with bounded derivatives A_{ij}. Similarly we can write $b_i^{m+1} - b_i^m$ in the form $B_i(\sigma^m, \sigma_\theta^m, \sigma_{\theta\theta}^m, \sigma_t^m)$, for some smooth function B_i. In other words f_m° takes the form

$$f_m^\circ = A_{ij}(\sigma^m, \sigma_\theta^m)\bar{u}_{ij}^{m+1} + a_{ij}^m v_{ij}^{m+1} + B_i(\sigma^m, \sigma_\theta^m, \sigma_{\theta\theta}^m, \sigma_t^m)\bar{u}_i^{m+1} + b_i^m v_i^{m+1}$$

In order to write h_m° in a similar form, note that

$$\begin{aligned} -\Xi^m + \Xi^{m-1} &= -\big(1 - \frac{|g^m|}{(R^m)^2}\big)[\bar{u}_n^{m+1}]_-^+ + \big(1 - \frac{|g^{m-1}|}{(R^{m-1})^2}\big)[\bar{u}_n^m]_-^+ \\ &= -[v_n^{m+1}]_-^+\big(1 - \frac{|g^m|}{(R^m)^2}\big) + [\bar{u}_n^m]_-^+\big(\frac{|g^m|}{(R^m)^2} - \frac{|g^{m-1}|}{(R^{m-1})^2}\big) \end{aligned}$$

and hence

$$h_m^\circ = -f_t^m\big(\frac{R^m}{|g^m|} - \frac{R^{m-1}}{|g^{m-1}|}\big) - \epsilon f_{\theta^4 t}^m\big(\frac{1}{|g^m|} - \frac{1}{|g^{m-1}|}\big) - [v_n^{m+1}]_-^+\big(1 - \frac{|g^m|}{(R^m)^2}\big) + [\bar{u}_n^m]_-^+\big(\frac{|g^m|}{(R^m)^2} - \frac{|g^{m-1}|}{(R^{m-1})^2}\big). \tag{3.313}$$

Similar structure is also shared by G_m°. The conclusion is that f_m°, G_m° and h_m° have a quadratic structure, always containing one term that can be bounded by the norms of $v^{j+1} - v^j$ and $\sigma^{j+1} = \sigma^j$, and the other term bounded by the norms of $\bar{u}^j$ and f^j (and hence a-priori bounded due to Lemma 3.2). In order to bound the first modes of $\sigma_{t^k}^{m+1}$, we shall use relation (3.303). If we rewrite (3.303) in the form

$$\sigma_t^{m+1} + \epsilon\sigma_{\theta^4 t}^{m+1} = -|g^m|[v_n^{m+1}]_-^+ + |g^m|h_m^\circ - f^m\sigma_t^{m+1},$$

upon applying ∂_{t^q}, multiplying the whole equation by $\sigma_{t^{q+1}}^{m+1}$ and integrating over $\mathbb{S}^1$, we obtain

$$\begin{aligned} &||\sigma_{t^{q+1}}^{m+1}||_{L^2}^2 + \epsilon||\sigma_{\theta^2 t^{q+1}}^{m+1}||_{L^2}^2 \le \Big(||(|g^m|[v_n^{m+1}]\circ\phi^m)_{t^q}||_{L^2} + ||(|g^m|h_m^\circ)_{t^q}||_{L^2} + ||(f^m\sigma_t^{m+1})_{t^q}||_{L^2}\Big)||\sigma_{t^{q+1}}^{m+1}||_{L^2} \\ &\le \frac{5}{4}\Big(||(|g^m|[v_n^{m+1}]\circ\phi^m)_{t^q}||_{L^2}^2 + |||g^m|h_m^\circ||_{L^2}^2 + ||(f^m\sigma_t^{m+1})_{t^q}||_{L^2}^2\Big) + \frac{1}{5}||\sigma_{t^{q+1}}^{m+1}||_{L^2}^2. \end{aligned} \tag{3.314}$$

We note that $||(|g^m|[v_n^{m+1}]\circ\phi^m)_{t^q}||_{L^2}^2 \le \frac{C}{\lambda}e^{m+1} + \lambda||\nabla v^{m+1}||_{H^1(\Omega)}^2$. It is further easy to see that $||(|g^m|h_m^\circ)_{t^q}||_{L^2}^2 + ||(f^m\sigma_t^{m+1})_{t^q}||_{L^2}^2 \le C\theta\sum_{k=0}^{l-1}\big(||\sigma_{t^k}^m||_{L^2}^2 + ||\sigma_{t^{k+1}}^{m+1}||_{L^2}^2\big)$. Thus, using (3.314), we obtain the bound

$$\begin{aligned} &\sum_{k=0}^{l-1}\Big\{\sup_{0\le s\le t}||\mathbb{P}_1\sigma_{t^k}^{m+1}||_{L^2}^2 + \int_0^t ||\mathbb{P}_1\sigma_{t^{k+1}}^{m+1}||_{L^2}^2\Big\} \le \frac{C}{\lambda}t\sup_{0\le s\le t}\bar{e}^{m+1} + \lambda\int_0^t \mathfrak{d}^{m+1} \\ &+C\theta\sum_{k=0}^{l-1}\Big\{\sup_{0\le s\le t}||\sigma_{t^k}^m||_{L^2}^2 + \int_0^t ||\sigma_{t^{k+1}}^m||_{L^2}^2\Big\}. \end{aligned} \tag{3.315}$$

Motivated by (3.315), we introduce the following temporal energy quantities:

$$e^j := \bar{e}^j + \sum_{k=0}^{l-1}||\mathbb{P}_1\sigma_{t^k}^j||_{L^2}^2; \quad d^j := \bar{d}^j + \sum_{k=0}^{l-1}||\mathbb{P}_1\sigma_{t^{k+1}}^j||_{L^2}^2. \tag{3.316}$$

Remark. Observe that the energy norm e^j is not equivalent to $\mathcal{E}^j$, because the first modes of σ^j are also controlled by $\mathcal{E}^j$; in other words $e^j \ge C_E||\mathbb{P}_1\sigma^j||_{L^2}^2$ for some constant C_E. An analogous remark holds for d^j.

The next lemma is the analogue of Lemma 3.2, where we collect preparatory statements, necessary to establish (3.318). Note that in the following the constant C_ϵ depends on ϵ and we shall make use of the ϵ-regularization (3.85).

Lemma 3.6 *(a) For any $j \in \mathbb{N}$, the following conservation law holds:*

$$\partial_t\Big\{\int_{\Omega^j} v^{j+1} + \int_{\mathbb{S}^1}\sigma^{j+1}\Big\} = -\int_{\mathbb{S}^1} f^j\sigma_t^{j+1} + \int_{\mathbb{S}^1} h_j^\circ.$$

(b) *The energy norms e^j and d^j are equivalent to $\mathcal{E}_{\|}+||\mathbb{P}_1\sigma||^2_{L^2}$ and $\mathcal{D}_{\|}$ respectively.*

(c) *There exist constants K^* and $0<\nu^*\leq 1$ such that for any $j\in\mathbb{N}$ the following inequality for space-time energies holds:*

$$\sup_{0\leq s\leq t}\mathfrak{e}^{j+1,\nu^*}+\int_0^t\mathfrak{d}^{j+1,\nu^*}\leq C_\epsilon(\theta+t)\big(\sup_{0\leq s\leq t}\mathfrak{e}^{j,\nu^*}+\int_0^t\mathfrak{d}^{j,\nu^*}\big)+K^*\int_0^t d^{j+1}(\tau)\,d\tau. \tag{3.317}$$

(d) *For a possibly smaller t^ϵ there exists a C such that for any $t\leq t^\epsilon$ the following inequality for the temporal energies holds:*

$$\sup_{0\leq s\leq t}e^{m+1}(s)+\int_0^t d^{m+1}(\tau)\,d\tau\leq C_\epsilon(\theta+t)\Big(\sup_{0\leq s\leq t}e^m(t)+\int_0^t d^m(\tau)\,d\tau\Big). \tag{3.318}$$

Sketch of the proof. The proof of part (a) is identical to the proof of (3.97) and follows by simply integrating (3.300) over Ω^m. The proof of part (b)is analogous to the proof of (3.114) presented in Subsection 3.2.1. It relies on part (a) of the lemma.
Proof of part (c). To prove (3.317), we proceed in the same way as in Subsection 3.2.5. For any pair of indices (μ,c) such that $|\mu|+2c\leq 2l-2$, we apply the differential operator ∂_c^μ to the equation (3.300), multiply with $-\Delta\partial_c^\mu v^{m+1}$ and integrate over $\Omega_{\mathbb{S}^1}$. We then carry out the same reduction procedure as in Subsection 3.2.5 (which in fact becomes simpler because the boundary $\mathbb{S}^1$ is time-independent). The estimates carry over analogously and the only terms where additional care is needed are the formally new expressions that involve f°_m, G°_m and h°_m. To illustrate this point, note that for instance, the analogue of the expression (3.242) in this case is simply given by

$$\begin{aligned}\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p+1}v^{m+1}\partial_c^{2p+1}[v_n^{m+1}]^+_-=&\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p+1}\big(-\sigma^m-\frac{\sigma^m_{\theta^2}}{R^m|g^m|}\big)\partial_c^{2p+1}\big(-\frac{\sigma_t^{m+1}R^m}{|g^m|}-\epsilon\frac{\sigma^{m+1}_{\theta^4t}}{|g^m|}\big)\\+\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p+1}v^{m+1}\partial_c^{2p+1}h^\circ_m+&\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p+1}G^\circ_m\partial_c^{2p+1}[v_n^{m+1}]^+_--\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p+1}G^\circ_m\partial_c^{2p+1}h^\circ_m.\end{aligned} \tag{3.319}$$

Observe that the first integral on RHS above will render an energy contribution to $\mathfrak{e}^{m+1}$, as it is explained in the energy identities from Section 2. However, the other three terms are formally new. Let us prove the energy estimate for the second term on RHS of (3.319): $\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p+1}v^{m+1}\partial_c^{2p+1}h^\circ_m$. First, we integrate by parts and use the Cauchy-Schwarz inequality:

$$\Big|\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p+1}v^{m+1}\partial_c^{2p+1}h^\circ_m\Big|=\Big|\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p}\partial_{\theta\theta}v^{m+1}\partial_c^{2p}h^\circ_m\Big|\leq\int_0^t||\partial_c^{2p}\partial_{\theta\theta}v^{m+1}||^2_{L^2}+\int_0^t||\partial_c^{2p}h^\circ_m||^2_{L^2}. \tag{3.320}$$

In order to estimate $||\partial_c^{2p}h^\circ_m||_{L^2}$ (recall that h°_m is given in the third line after (3.303)), we use the inequality (3.252) (to estimate $\epsilon f^m_{\theta^4t}$), the quadratic structure of h°_m as given in (3.313) and the a-priori estimate $E^j_{\alpha,\nu}+\int_0^t D^j_{\alpha,\nu}\leq\theta$ for $j=m,m+1$. We obtain

$$||\partial_c^{2p}h^\circ_m||^2_{L^2}\leq C\mathfrak{e}^m\big(\mathfrak{D}^m+\frac{C}{\epsilon^4}\mathfrak{E}^m\big)+C\mathfrak{E}^m\mathfrak{d}^{m+1}+C\mathfrak{D}^m\mathfrak{e}^m. \tag{3.321}$$

On the other hand, using the boundary condition (3.301) for v^{m+1} and the definition of $\mathfrak{e}^j$, we obtain $||\partial_c^{2p}\partial_{\theta\theta}v^{m+1}||^2_{L^2}\leq\frac{C}{\epsilon}\mathfrak{e}^m$. Using this, the estimates (3.321) and (3.320), we obtain

$$\begin{aligned}\Big|\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p+1}v^{m+1}\partial_c^{2p+1}h^\circ_m\Big|&\leq C\big(\frac{t}{\epsilon}+\int_0^t\mathcal{D}^m+\frac{t}{\epsilon^4}\sup_{0\leq s\leq t}\mathfrak{E}^m\big)\sup_{0\leq s\leq t}\mathfrak{e}^m+C\sup_{0\leq s\leq t}\mathfrak{E}^m\int_0^t\mathfrak{d}^{m+1}\\&\leq C(\nu,\alpha)\big(\frac{t}{\epsilon}+\theta+\frac{t\theta}{\epsilon^4}\big)\sup_{0\leq s\leq t}\mathfrak{e}^m+CL\int_0^t\mathfrak{d}^{m+1}.\end{aligned}$$

Using the definition of G°_m (given in the second equation after (3.303)) and the the definition of $\mathfrak{e}^j$, we easily obtain $||\partial_c^{2p+2}G^\circ_m||^2_{L^2}\leq\frac{C}{\epsilon}\mathfrak{e}^m$. Using the integration by parts, the trace inequality and the previous estimate,

we bound the third term on RHS of (3.319):

$$\begin{aligned}&\Big|\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p+1}G_m^\circ\partial_c^{2p+1}[v_n^{m+1}]_-^+\Big|=\Big|\int_0^t\int_{\mathbb{S}^1}\partial_c^{2p+2}G_m^\circ\partial_c^{2p}[v_n^{m+1}]_-^+\Big|\\&\le\frac{C}{\lambda}\int_0^t||\partial_c^{2p+2}G_m^\circ||_{L^2}^2+\lambda\int_0^t||\partial_c^{2p}[v_n^{m+1}]_-^+||_{L^2(\mathbb{S}^1)}^2\le\frac{Ct}{\lambda\epsilon}\sup_{0\le s\le t}\mathfrak{e}^m+\lambda\int_0^t\mathfrak{d}^{m+1}.\end{aligned}$$

Similarly we can deal with the last integral on RHS of (3.319). Note that we can afford to have constants in our estimates that contain ϵ in the denominator. This fact allowed us to use the jump relation (3.303) to estimate the highest order terms above. This is in fact analogous to the proof of uniqueness in Section 5 from [21]. Having obtained the energy estimates, we can conclude (3.317) in a way analogous to the derivation of (3.286) starting from (3.284). Here we also exploit the fact that $\sigma^j(0)=v^j(0)=0$ for any $j\in\mathbb{N}$.

Proof of part (d). The inequality (3.318) follows by estimating RHS of the identity (3.311), using (3.315) and Lemma 3.6. These estimates are, indeed, analogous to the estimates from Subsection 3.2.3. The only novelty are the formally new terms $\partial_{t^k}f_m^\circ$, $\partial_{t^k}G_m^\circ$ and $\partial_{t^k}h_m^\circ$ appearing in the definitions (3.307), (3.308) and (3.309) of f_m', G_m' and h_m' respectively, but they can be dealt with analogously to the proof of part (c) of Lemma 3.6. □

Using (3.318) together with part (c) of Lemma 3.6 yields a contraction bound on the total energy:

$$\begin{aligned}&\sup_{0\le s\le t}e^{m+1}(s)+\int_0^t d^{m+1}(\tau)\,d\tau+\sup_{0\le s\le t}\mathfrak{e}^{m+1,\nu^*}+\int_0^t\mathfrak{d}^{m+1,\nu^*}\\&\le\Lambda\Big(\sup_{0\le s\le t}e^m(t)+\int_0^t d^m(\tau)\,d\tau\Big)+C(\theta+t)\Big(\sup_{0\le s\le t}\mathfrak{e}^{m,\nu^*}+\int_0^t\mathfrak{d}^{m,\nu^*}\Big)+K^*\Lambda\Big(\sup_{0\le s\le t}e^m(t)+\int_0^t d^m(\tau)\,d\tau\Big)\\&\le\Lambda^*\Big(\sup_{0\le s\le t}e^m(s)+\int_0^t d^m(\tau)\,d\tau+\sup_{0\le s\le t}\mathfrak{e}^{m,\nu^*}+\int_0^t\mathfrak{d}^{m,\nu^*}\Big),\end{aligned}\tag{3.322}$$

where $\Lambda^*:=\Lambda(1+K^*)+C(\theta+t)$. Note that $\Lambda^*<1$, if we choose θ small enough and possibly a smaller t^ϵ so that $\Lambda<\frac{1-C(\theta+t)}{1+K^*}$. Having proved (3.322), we conclude that the sequence $(v^k,\sigma^k)_{k\in\mathbb{N}}$ is a Cauchy sequence in the energy space defined for some t^ϵ small enough. Passing to the limit we recover the solution to the regularized Stefan problem on the time interval $[0,t^\epsilon[$.

3.3.2 Uniqueness

We shall use Gronwall's lemma to prove uniqueness of the solutions to the regularized Stefan problem:

Lemma 3.7 *In the class of functions u,f satisfying $\sup_{0\le s\le t^\epsilon}E_{\alpha,\nu}(u,f)+\int_0^{t^\epsilon}D_{\alpha,\nu}(u,f)(s)\,ds\le L$ (where L may be chosen smaller if necessary), the solution (u,f) to the regularized Stefan problem is unique.*

Proof. Let us assume that there exists another solution $(\tilde u,\tilde f)$ satisfying the same initial condition $(\tilde u(x,0),\tilde f(\theta,0))=(u_0(x),f_0(\theta))$ and the same smallness bound $\sup_{0\le s\le t^\epsilon}E_{\alpha,\nu}(\tilde u,\tilde f)+\int_0^{t^\epsilon}D_{\alpha,\nu}(\tilde u,\tilde f)(s)\,ds\le L$. We shall formulate the Stefan problem on the fixed domain by pulling both functions u and $\tilde u$ onto the fixed domain $\Omega_{S_1(x_0,0)}$. Without loss of generality we assume $\Omega_{S_1(x_0,0)}=\Omega_{\mathbb{S}^1}$. To this end, we shall apply the change of variables described in Subsection 3.3.1. We define $\Theta(t,x):=\pi(t,x)x$ and $\tilde\Theta(t,x):=\tilde\pi(t,x)x$, where π and $\tilde\pi$ are defined as in (3.289) by substituting R^{m-1} with $(1+f)$ and $(1+\tilde f)$ respectively. We set $U:=u\circ\Theta$ and $\tilde U=\tilde u\circ\tilde\Theta$. The corresponding coefficients a_{ij}, b_i, $\tilde a_{ij}$, $\tilde b_i$ are defined analogously to a_{ij}^m, b_i^m (given by (3.291)). Subtracting the two sets of equations (that we get for (U,f) and $(\tilde U,\tilde f)$), we obtain a new problem for $(V,h):=(U-\tilde U,f-\tilde f)$ $\big($note that $\tilde R=1+\tilde f$ and $|\tilde g|=\sqrt{\tilde R^2+\tilde f_\theta^2}\big)$:

$$V_t-\Delta V=A^\circ,\tag{3.323}$$

$$V=-h-\frac{h_{\theta\theta}}{R|g|}+B^\circ,\tag{3.324}$$

$$[V_n]_-^+=-\frac{h_tR}{|g|}-\epsilon\frac{h_{\theta^4t}}{|g|}+C^\circ\ \text{ on }\mathbb{S}^1,\tag{3.325}$$

where

$$A^\circ = (a_{ij} - \tilde{a}_{ij})U_{ij} + \tilde{a}_{ij}V_{ij} + (b_i - \tilde{b}_i)U_i + \tilde{b}_i V_i,$$

$$B^\circ = N^*(f) - N^*(\tilde{f}) + \tilde{f}_{\theta\theta}\Big(\frac{1}{R|g|} - \frac{1}{\tilde{R}|\tilde{g}|}\Big),$$

$$C^\circ = -\tilde{f}_t\Big(\frac{R}{|g|} - \frac{\tilde{R}}{|\tilde{g}|}\Big) - \epsilon \tilde{f}_{\theta^4 t}\Big(\frac{1}{|g|} - \frac{1}{|\tilde{g}|}\Big) - \Xi(f) + \Xi(\tilde{f}).$$

Here $N^*(f)$ is defined in line after (3.96) and $\Xi(f)$ and $\Xi(\tilde{f})$ are defined by replacing f^m by f and $\tilde{f}$ in the definition of $\Xi(f^m)$ in the line after (3.299), respectively. We use the notation from (2.62) - (2.66) and Lemma 2.2 from Section 2 to derive the accompanying energy identities. To this end, we set

$$\mathcal{F} = A^\circ, F \equiv 1, G = B^\circ, h = C^\circ, \mathcal{U} = V, \omega = \chi = h \text{ and } \psi = R.$$

Same remark as in the line after (3.310) applies here; since $\Gamma = \mathbb{S}^1$ is time-independent, we conclude that the expressions $O(\mathcal{U})$ and $P(\mathcal{U})$ defined in (2.73) and (2.74) are equal to 0. In formal comparison to the problem (3.300) - (3.303), f takes the role of f^{m+1}, and $\tilde{f}$ the role of f^m. Additionally, the cross-terms vanish since $\omega = \chi = h$. We are naturally led to the following energy quantities:

$$\mathcal{E}^* := \tilde{\mathcal{E}}_\epsilon(V,h), \quad \mathcal{D}^* := \tilde{\mathcal{D}}_\epsilon(V,h),$$

where $\tilde{\mathcal{E}}_\epsilon$ and $\tilde{\mathcal{D}}_\epsilon$ are defined by (2.67) and (2.68), respectively. In addition to this, we define $Q^* = Q(h,h,R)$ and analogously S^*. Using the identity (2.71), we obtain

$$\mathcal{E}^*(t) + \int_0^t \mathcal{D}^*(s)\,ds = \int_0^t \int_{\Omega_{\mathbb{S}^1}} \big(V + V_t\big)A + \int_0^t \int_{\mathbb{S}^1} \big\{Q^* + S^*\big\}. \tag{3.326}$$

In addition to the above energies $\mathcal{E}^*$ and $\mathcal{D}^*$, we define

$$\mathfrak{E}^* = ||\nabla V||^2_{L^2(\Omega)} + \int_{\mathbb{S}^1} \Big\{|h_{\theta^2}|^2 - |h_\theta|^2\Big\} - \int_{\mathbb{S}^1} |\mathbb{P}_{2+} h_{\theta^2}|^2 \Big(\frac{1}{|g|^3} - 1\Big) + \epsilon \int_{\mathbb{S}^1} \Big\{|h_{\theta^4}|^2 - |h_{\theta^3}|^2\Big\} - \epsilon \int_{\mathbb{S}^1} |\mathbb{P}_{2+} h_{\theta^4}|^2 \Big(\frac{1}{R|g|^3} - 1\Big)$$

$$\mathfrak{D}^* = ||\nabla V||^2_{H^1(\Omega)} + \int_{\mathbb{S}^1} \Big\{|h_{\theta t}|^2 - |h_t|^2\Big\} + \epsilon \int_{\mathbb{S}^1} \Big\{|f_{\theta^3 t}|^2 - |f_{\theta^2 t}|^2\Big\}$$

Note that the quantities $\mathfrak{E}^*$ and $\mathfrak{D}^*$ are identical to $\mathfrak{E}$ and $\mathfrak{D}$ in the case when $l = 1$. Our goal is to prove that for some $0 < \alpha^* \le 1$ the following inequality holds

$$\mathcal{E}^*(t) + \alpha^* \mathfrak{E}^*(t) + \int_0^t \big\{\mathcal{D}^*(s) + \alpha^* \mathfrak{D}^*(s)\big\}\,ds \le C \int_0^t \big\{\mathcal{E}^*(s) + \mathfrak{E}^*(s)\big\}\,ds + C(\sqrt{L} + \alpha^*) \int_0^t \mathcal{D}^*(s)\,ds, \tag{3.327}$$

which would enable us to absorb the multiple of $\int_0^t \mathcal{D}^*$ on the right-hand side into the left-hand side and then use the Gronwall's inequality to conclude that $\mathcal{E}^*(t) + \mathfrak{E}^*(t) = 0$ for any $t \ge 0$. It is essential that the constant C in the above estimate does not depend on ϵ so that the smallness bound on L remains independent of ϵ. That the identity (3.327) indeed holds, follows analogously to the energy estimates from Subsections 3.2.5 and 3.2.3 applied to the right-hand side of (3.326). Here we strongly exploit the uniform bounds on $E_{\alpha,\nu}(u,f)$ and $E_{\alpha,\nu}(\tilde{u},\tilde{f})$. A major difference from the existence part of the proof is the absence of cross-terms in the energy identities (since $\omega = \chi$ in the notation of Section 2). In addition to that, we work in a lower order energy space ($l = 1$) and we can thus use the above uniform estimates to bound the term $[V_n]_-^+$ by $C\sqrt{L}$ in L^∞-norm. Knowing that the ϵ-dependence comes only from the estimates of the cross-terms (cf. (3.254) - (3.256), (3.279), (3.280), (3.192) - (3.194)), we conclude that the constants on the right-hand side of (3.327) *do not* depend on ϵ. Choosing L and α^* suitably small (3.327) implies $\mathcal{E}^*(t) + \mathfrak{E}^*(t) \le C \int_0^t \big\{\mathcal{E}^*(s) + \mathfrak{E}^*(s)\big\}\,ds$, which, by Gronwall's inequality leads to $\mathcal{E}^*(t) + \mathfrak{E}^*(t) = 0$. This finishes the proof of the uniqueness statement of Theorem 3.4. □

3.3.3 Continuity for the regularized Stefan problem

The continuity statement is important because it will allow us to extend the solution beyond the the time interval $[0,t^\epsilon[$.

Lemma 3.8 *The total energy* $E^\epsilon_{\alpha,\nu}(u,f)(t)+\int_0^t D^\epsilon_{\alpha,\nu}(u,f)(s)\,ds$ *is continuous as a function of* t *on* $[0,t^\epsilon[$.

Sketch of the proof. Integrating over $[s,t]$ instead of $[0,t]$ while arriving at the identity (3.113) in Section 3.1, we obtain

$$\mathcal{E}^{m+1}_\epsilon(t)-\mathcal{E}^{m+1}_\epsilon(s)+\int_s^t \mathcal{D}^{m+1}_\epsilon(\tau)\,d\tau=\int_s^t A(u^{m+1},f^m,f^{m+1})+\int_s^t\int_{\Gamma^m}\Big\{O^m+P^m\Big\}+\int_s^t\int_{\mathbb{S}^1}\Big\{Q^m+S^m\Big\}. \tag{3.328}$$

We claim that we may pass to the limit as $m\to\infty$ in each of the terms in (3.328) to obtain:

$$\mathcal{E}_\epsilon(t)-\mathcal{E}_\epsilon(s)+\int_s^t \mathcal{D}_\epsilon(\tau)\,d\tau=\int_s^t \bar{A}+\int_s^t\int_{\Gamma}\Big\{\bar{O}+\bar{P}\Big\}+\int_s^t\int_{\mathbb{S}^1}\Big\{\bar{Q}+\bar{S}\Big\}, \tag{3.329}$$

where

$$\bar{A}=A(u,f,f),\ \bar{O}=\sum_{k=1}^{l-1}O(u_{t^k}),\ \bar{P}=\sum_{k=0}^{l-1}P(u_{t^k}),\ \bar{Q}=\sum_{k=1}^{l-1}Q(u_{t^k},f_{t^k},R)\ \text{and}\ \bar{S}=\sum_{k=0}^{l-1}S(u_{t^k},f_{t^k},R).$$

The passage to the limit is justified for the following reason: we may apply the differential operator ∂_{t^k} to the equality (3.295) to express the derivatives of u^{m+1} in terms of derivatives of $\bar{u}^{m+1}$ and f^m. This enables us to express the L^2 norms of $\partial_{t^k}u^{m+1}$ on Ω^m in terms of the space-time norms of $\bar{u}^{m+1}$ and f^m on $\Omega_{\mathbb{S}^1}$ and $\mathbb{S}^1$, respectively. Hence, using the strong convergence of the sequence (u^j,f^j), we may pass to the limit to conclude (3.329). In order to bound RHS of (3.329), by the same argument, we pass to the limit as $m\to\infty$ in the estimate (3.199), where we keep in mind that we work on the time interval $[s,t]$. Note that all the estimates coming from the cross-terms will vanish, and those are precisely the ones involving the constant $\bar{C}$. As a consequence we obtain

$$\begin{aligned}&\Big|\int_s^t \bar{A}+\int_s^t\int_\Gamma \bar{O}+\bar{P}-\int_s^t\int_{\mathbb{S}^1}\bar{Q}+\bar{S}\Big|\le CC_0(t-s)\sup_{s\le\tau\le t}\big(\mathcal{E}(\tau)+\mathfrak{E}(\tau)+\theta_1+\int_s^t\mathfrak{D}\big)\\&+C(\lambda+\sup_{0\le\tau\le t}||\mathbb{P}_1 f||^2_{L^2}+C\sqrt{\mathfrak{E}})\int_s^t\mathcal{D}_\epsilon+C\big(\sqrt{\mathfrak{E}}+\lambda+(\mathcal{E}_{(0)})^{1/2}\big)\int_0^t\mathfrak{D}_\epsilon\end{aligned} \tag{3.330}$$

Using (3.329), (3.330) and the triangle inequality, we obtain for any $0\le s<t\le t^\epsilon$:

$$\begin{aligned}&\big|\mathcal{E}_\epsilon(t)-\mathcal{E}_\epsilon(s)\big|\le C\Big|\int_s^t\mathcal{D}_\epsilon(\tau)\,d\tau\Big|\Big(1+C(\lambda+\sup_{0\le\tau\le t}||\mathbb{P}_1 f||^2_{L^2}+\sqrt{\mathfrak{E}})\Big)\\&+CC_0(t-s)\sup_{s\le\tau\le t}\big(\mathcal{E}(\tau)+\mathfrak{E}(\tau)+\theta_1+\int_s^t\mathfrak{D}\big)+C\big(\sqrt{\mathfrak{E}}+\lambda+(\mathcal{E}_{(0)})^{1/2}\big)\int_s^t\mathfrak{D}_\epsilon\longrightarrow 0\quad\text{as}\quad s\to t,\end{aligned}$$

since $\sup_{0\le\tau\le t^\epsilon}\mathcal{E}_\epsilon(\tau)$ and $\mathfrak{E}_\epsilon$ are bounded on $[0,t^\epsilon[$. This proves the continuity of $\mathcal{E}_\epsilon$ on the small time interval $[0,t^\epsilon[$. In a completely analogous fashion, by passing to the limit in (3.284), it follows that $\mathfrak{E}_\epsilon$ is continuous on the interval $[0,t^\epsilon[$. □

3.3.4 Momentum conservation and smallness of the first modes

Recall that the smallness condition on $||f||_{H^{2l}}$, which is repeatedly used to close the estimates, is guaranteed through a t and ϵ-dependent estimate (3.136). It is thus not applicable if we want to prove that the solution exists on a uniform time interval independent of ϵ, nor is it useful for extending the solution globally because the bound grows as t becomes large. In order to overcome these difficulties, we shall prove a crucial 'conservation-of-momentum' law. Let $M\le L/2$, where L is given by Theorem 3.4, and let (u,f) be the unique solution of the regularized Stefan problem with the initial conditions satisfying $E_{\alpha,\nu}(u_0,f_0)\le M$.

Lemma 3.9 *Let $p_a(x,y)=x+\frac{R_*^2 x}{r^2}$ and $p_b(x,y)=y+\frac{R_*^2 y}{r^2}$. Then, on the time interval of existence of the solution to the regularized Stefan problem, the following conservation laws hold:*

$$\partial_t \int_\Omega u p_a = \partial_t \int_{\mathbb{S}^1} F_a(R,\theta)\,d\theta + \epsilon \int_{\mathbb{S}^1} \big(f_{\theta^4 t} R\cos\theta + R_*^2 f_{\theta^4 t}\frac{R\cos\theta+\alpha}{(R\cos\theta+\alpha)^2+(R\sin\theta)^2}\big)\,d\theta \tag{3.331}$$

and

$$\partial_t \int_\Omega u p_b = \partial_t \int_{\mathbb{S}^1} F_b(R,\theta)\,d\theta + \epsilon \int_{\mathbb{S}^1} \big(f_{\theta^4 t} R\sin\theta + R_*^2 f_{\theta^4 t}\frac{R\sin\theta}{(R\cos\theta+\alpha)^2+(R\sin\theta)^2}\big)\,d\theta, \tag{3.332}$$

where F_a and F_b are defined by (1.25) and (1.26) respectively.

Proof. Note that both p_a and p_b are harmonic away from the origin $r=0$ and enjoy the property

$$\partial_n p_a\big|_{\partial\Omega} = \partial_n p_b\big|_{\partial\Omega} = 0. \tag{3.333}$$

Let further $\Omega_\delta := \Omega \setminus B_\delta(0)$. Multiply (1.1) by p_a and integrate over Ω_δ. Using the Green's second identity, harmonicity of p_a and (3.333), we obtain

$$\int_{\Omega_+} u_t^+ p_a - \int_\Gamma \partial_n u^+ p_a\,dS + \int_\Gamma u^+ \partial_n p_a = 0,$$

$$\int_{\Omega_-} u_t^- p_a + \int_\Gamma \partial_n u^- p_a\,dS - \int_\Gamma u^- \partial_n p_a - \int_{S_\delta(0)} u_n p_a + \int_{S_\delta(0)} u\partial_n p_a = 0.$$

Summing the above equations and using the jump condition (1.3), we obtain

$$\partial_t \int_{\Omega_\delta} u p_a = \int_\Gamma V p_a\,dS + \int_{S_\delta(0)} u_n p_a - \int_{S_\delta(0)} u\partial_n p_a.$$

Note that $\partial_n p_a|_{S_\delta(0)} = \partial_r p_a|_{S_\delta(0)} = \cos\theta - R_*^2\frac{\cos\theta}{\delta^2}$. Hence

$$\int_{S_\delta(0)} u\partial_n p_a\,dS = \int_{\mathbb{S}^1} u(\delta,\theta)\big(\cos\theta - R_*^2\frac{\cos\theta}{\delta^2}\big)\delta\,d\theta = \int_{\mathbb{S}^1} u(\delta,\theta)\delta\cos\theta - R_*^2\int_{\mathbb{S}^1} u(\delta,\theta)\frac{\cos\theta}{\delta}\,d\theta.$$

Note that the first integral on right-most side obviously converges to 0 as $\delta\to 0$. As to the second integral, observe that

$$\int_{\mathbb{S}^1} u(\delta,\theta)\frac{\cos\theta}{\delta}\,d\theta = \int_{\mathbb{S}^1} \frac{u(\delta,\theta)-u(0,0)}{\delta}\cos\theta\,d\theta \to \int_{\mathbb{S}^1} u_r(0,0)\cos\theta\,d\theta = 0 \quad \text{as } \delta\to 0.$$

Similarly $\int_{S_\delta(0)} u_n p_a \to 0$ as $\delta\to 0$. Finally, passing to the limit as $\delta\to 0$, we get

$$\partial_t \int_\Omega u p_a = \int_\Gamma V p_a\,dS, \tag{3.334}$$

where the integral $\int_\Omega u p_a$ is interpreted in the sense of the existing limit as $\delta\to 0$, i.e. $\int_\Omega u p_a = \lim_{\delta\to 0}\int_{\Omega_\delta} u p_a$ (keep in mind that $p_a \approx \frac{1}{r}$ at the origin and this singularity is integrable). Upon abbreviating $R_{x_0}^2 = (R\cos\theta + x_0)^2 + (R\sin\theta)^2$ and recalling the definition (1.25) of F_a, we get

$$\begin{aligned}\int_\Gamma V p_a &= \int_{\mathbb{S}^1} \big(\frac{f_t R}{|g|} + \epsilon\frac{f_{\theta^4 t}}{|g|}\big)\big(R\cos\theta + x_0 + R_*^2\frac{R\cos\theta+x_0}{R_{x_0}^2}\big)|g|\,d\theta \\ &= \int_{\mathbb{S}^1} f_t\partial_R F_a(R,\theta) + \epsilon\int_{\mathbb{S}^1}\big(f_{\theta^4 t}R\cos\theta + R_*^2 f_{\theta^4 t}\frac{R\cos\theta+x_0}{R_{x_0}^2}\big)\,d\theta \\ &= \partial_t\int_{\mathbb{S}^1}\{F_a(R,\theta)-F_a(1,\theta)\} + \epsilon\int_{\mathbb{S}^1}\big(f_{\theta^4 t}R\cos\theta + R_*^2 f_{\theta^4 t}\frac{R\cos\theta+x_0}{R_{x_0}^2}\big)\,d\theta,\end{aligned} \tag{3.335}$$

and this, together with (3.334) proves the first claim of the lemma. The second claim follows in a fully analogous way. □

Remark. One could conceivably try to obtain a hierarchy of conservation laws for higher momenta, by multiplying the equation (1.10) by appropriate functions that restrict to a combination of cosines and sines of integer multiples of θ on $\mathbb{S}^1$. For instance, in order to control second momenta, a natural choice for the test function p is $p(x,y)=y^2-x^2+R_*^4\frac{y^2-x^2}{r^4}$ since p is harmonic, $\partial_n p|_{\partial\Omega}=0$ and $p|_\Gamma=-R^2\cos 2\theta-R_*^4\frac{\cos 2\theta}{R^2}$. However, this strategy fails to render new conservation laws, since p contains a singularity of order $1/r^2$, which fails to be integrable.

We shall exploit the conservation laws given by (1.22) and Lemma 3.9 to prove that the natural energy $\mathcal{E}_{(0)}$ (given by (1.29)) is in fact positive and controls all the modes of f. To this end, the following lemma is necessary:

Lemma 3.10 *There exists a constant C such that the following inequalities hold:*

$$|a_1|+|b_1|\le C\Big(|m_0|+|m_a|+|m_b|+C||\nabla u||_{L^2(\Omega^\pm)}+C\sum_{k\ge 2}(|a_k|+|b_k|)+C\epsilon\sqrt{t}\Big(\int_0^t ||f_t||^2_{L^2}\Big)^{1/2}\Big) \tag{3.336}$$

$$|a_0|\le C\Big(|m_0|+|m_a|+|m_b|+||u||_{L^2(\Omega^\pm)}+||\nabla u||_{L^2(\Omega^\pm)}+\sum_{k\ge 2}(|a_k|+|b_k|)+\epsilon t\int_0^t ||f_t||^2_{L^2}\Big). \tag{3.337}$$

Proof. Recall that $m_i=\int_\Omega u_0 p_i+\int_{\mathbb{S}^1}F_i(R_0,\theta)$ for $i=a,b$ and m_0 is given by (1.27). Observe that

$$\int_{\mathbb{S}^1}F_a(1+f,\theta)-F_a(1,\theta)\,d\theta=\int_{\mathbb{S}^1}\partial_f F_a(1,\theta)\,d\theta+\int_{\mathbb{S}^1}O(f^2). \tag{3.338}$$

From the definition (1.25) of F_a we have $\partial_f F_a(1,\theta)=\cos\theta+x_0+R_*^2\frac{\cos\theta+x_0}{(\cos\theta+x_0)^2+\sin\theta^2}$. Using the decomposition $f=\sum_{k=0}a_k\cos k\theta+b_k\sin k\theta$ and abbreviating $G_a(R,\theta):=f_{\theta^4 t}R\cos\theta+R_*^2 f_{\theta^4 t}\frac{R\cos\theta+x_0}{R^2_{x_0}}$, we conclude

$$\begin{aligned}&a_1\int_{\mathbb{S}^1}\big(\cos^2\theta+x_0\cos\theta+R_*^2\frac{\cos^2\theta+x_0\cos\theta}{1+2x_0\cos\theta+x_0^2}\big)\,d\theta+b_1\int_{\mathbb{S}^1}\sin\theta(\cos\theta+x_0)+R_*^2\frac{\sin\theta\cos\theta+x_0\sin\theta}{1+2x_0\cos\theta+x_0^2}\,d\theta\\&=\int_{\mathbb{S}^1}F_a(1+f,\theta)-F_a(1,\theta)\,d\theta-\int_{\mathbb{S}^1}O(f^2)-\int_{\mathbb{S}^1}\partial_f F_a(1,\theta)(a_0+\sum_{k\ge 2}a_k\cos k\theta+b_k\sin k\theta)+\epsilon\int_{\mathbb{S}^1}G_a(R,\theta)\,d\theta.\end{aligned} \tag{3.339}$$

Observe that the second integral on LHS above vanishes since $\frac{\sin\theta\cos\theta+x_0\sin\theta}{1+2x_0\cos\theta+x_0^2}$ is an odd function. On the other hand, the following claim holds:
Claim:

$$\int_{\mathbb{S}^1}\big(\cos^2\theta+x_0\cos\theta+R_*^2\frac{\cos^2\theta+x_0\cos\theta}{1+2x_0\cos\theta+x_0^2}\big)\,d\theta=(1+R_*^2)\pi. \tag{3.340}$$

Proof of the claim: Obviously $\int_{\mathbb{S}^1}\big(\cos^2\theta+x_0\cos\theta\big)\,d\theta=\pi$. To prove the claim it suffices to show that

$$\int_0^{2\pi}\frac{\cos^2\theta+x_0\cos\theta}{1+x_0^2+2x_0\cos\theta}\,d\theta=\pi.$$

We first compute

$$\begin{aligned}\int_0^{2\pi}\frac{\cos^2\theta}{1+x_0^2+2x_0\cos\theta}\,d\theta&=\frac{1}{2x_0}\int_0^{2\pi}\frac{\Big\{\cos\theta+\frac{1+x_0^2}{2x_0}\Big\}^2-\frac{1+x_0^2}{x_0}\cos\theta-\frac{(1+x_0^2)^2}{4x_0^2}}{\frac{1+x_0^2}{2x_0}+\cos\theta}\,d\theta\\&=\frac{1}{2x_0}\int_0^{2\pi}\{\frac{1+x_0^2}{2x_0}+\cos\theta\}\,d\theta-\frac{1+x_0^2}{2x_0^2}\int_0^{2\pi}\frac{\cos\theta+\frac{1+x_0^2}{2x_0}-\frac{1+x_0^2}{4x_0}}{\frac{1+x_0^2}{2x_0}+\cos\theta}\,d\theta\\&=\frac{\{1+x_0^2\}2\pi}{4x_0^2}-\frac{\{1+x_0^2\}2\pi}{2x_0^2}+\frac{\{1+x_0^2\}^2}{4x_0^2}\int_0^{2\pi}\frac{1}{1+x_0^2+2x_0\cos\theta}\,d\theta\\&=-\frac{\{1+x_0^2\}2\pi}{4x_0^2}+\frac{\{1+x_0^2\}^2}{4x_0^2}\int_0^{2\pi}\frac{1}{1+x_0^2+2x_0\cos\theta}\,d\theta.\end{aligned}$$

By a standard integration formula,

$$\int_0^{2\pi} \frac{1}{1+x_0^2+2x_0\cos\theta}\,d\theta = \frac{2}{\sqrt{\{1+x_0^2\}^2-4x_0^2}}\arctan\{\frac{\sqrt{\{1+x_0^2\}^2-4x_0^2}}{1+x_0^2+2x_0}\tan\frac{\theta}{2}\}|_{-\pi}^{\pi} = \frac{2\pi}{1-x_0^2}.$$

We note that in the stable vase, $|x_0|\le\sqrt{2}-1<1$. We hence deduce

$$\int_0^{2\pi}\frac{\cos^2\theta}{1+x_0^2+2x_0\cos\theta}\,d\theta = -\frac{\{1+x_0^2\}2\pi}{4x_0^2}+\frac{\{1+x_0^2\}^2 2\pi}{4x_0^2(1-x_0^2)} = \frac{\{1+x_0^2\}2\pi}{4x_0^2}\{\frac{1+x_0^2}{1-x_0^2}-1\} = \frac{\{1+x_0^2\}\pi}{1-x_0^2}.$$

On the other hand

$$\begin{aligned}&\int_0^{2\pi}\frac{x_0\cos\theta}{1+x_0^2+2x_0\cos\theta}\,d\theta = \int_0^{2\pi}\frac{\{x_0\cos\theta+\frac{1+x_0^2}{2}\}-\frac{1+x_0^2}{2}}{1+x_0^2+2x_0\cos\theta}\,d\theta\\ &=\pi-\frac{1+x_0^2}{2}\int_0^{2\pi}\frac{1}{1+x_0^2+2x_0\cos\theta}\,d\theta = \pi-\frac{1+x_0^2}{2}\frac{2\pi}{1-x_0^2} = -\frac{2x_0^2\pi}{1-x_0^2}.\end{aligned}$$

From here, we obtain

$$\int_0^{2\pi}\frac{\cos^2\theta+x_0\cos\theta}{1+x_0^2+2x_0\cos\theta}\,d\theta = \frac{\{1+x_0^2\}\pi}{1-x_0^2}-\frac{2x_0^2\pi}{1-x_0^2} = \pi.$$

and this finishes the proof of the claim. □

Using the fact that $\int_{\mathbb{S}^1} F_a(1+f,\theta)-F_a(1,\theta)\,d\theta = m_a-\int_\Omega up_a$, we deduce from (3.339) and (3.340)

$$(1+R_*^2)\pi a_1 = m_a-\int_\Omega up_a-\int_{\mathbb{S}^1}O(f^2)-\int_{\mathbb{S}^1}\partial_f F_a(1,\theta)(a_0+\sum_{k\ge2}a_k\cos k\theta+b_k\sin k\theta)+\epsilon\int_{\mathbb{S}^1}G_a(R,\theta)\,d\theta. \tag{3.341}$$

In a fully analogous way, we obtain

$$(1+R_*^2)\pi b_1 = m_b-\int_\Omega up_b-\int_{\mathbb{S}^1}O(f^2)-\int_{\mathbb{S}^1}\partial_f F_b(1,\theta)(a_0+\sum_{k\ge2}a_k\cos k\theta+b_k\sin k\theta)+\epsilon\int_{\mathbb{S}^1}G_b(R,\theta)\,d\theta. \tag{3.342}$$

Note that for $i=a,b$, we have

$$\begin{aligned}&\Big|\int_{\mathbb{S}^1}O(f^2)\Big|+\Big|\int_{\mathbb{S}^1}\partial_f F_i(1,\theta)(a_0+\sum_{k\ge2}a_k\cos k\theta+b_k\sin k\theta)\Big| \le C(\sum_{k\ge0}a_k^2+b_k^2+|a_0|+\sum_{k\ge2}|a_k|+|b_k|)\\ &\le C\sqrt{M}(|a_1|+|b_1|)+C(1+\sqrt{M})\sum_{k\ne1}(|a_k|+|b_k|).\end{aligned}$$

In order to bound the second terms on RHS of (3.341) and (3.342) respectively, we use the polar coordinates:

$$\begin{aligned}&\int_\Omega up_a = \int_0^{R_*}\int_{\mathbb{S}^1}u(r,\theta)(r+\frac{R_*^2}{r})\cos\theta r\,dr\,d\theta = \int_0^{R_*}\int_{\mathbb{S}^1}u(r,\theta)(r^2+R_*^2)\cos\theta\,dr\,d\theta\\ &=-\int_0^{R_*}\int_{\mathbb{S}^1}u_\theta(r,\theta)(r^2+R_*^2)\sin\theta\,dr\,d\theta = -\int_0^{R_*}\int_{\mathbb{S}^1}\big(-u_x r\sin\theta+u_y r\cos\theta\big)(r^2+R_*^2)\sin\theta\,dr\,d\theta\\ &\le(R_*^2+R_*^2)\int_0^{R_*}\int_{\mathbb{S}^1}\big(|u_x|+|u_y|\big)r\,dr\,d\theta \le C||\nabla u||_{L^2(\Omega)},\end{aligned}$$

and analogously for $\int_\Omega up_b$. Using the integration by parts, boundedness of $||f||_{H^l}$ on $[0,T]$ and the Cauchy-Schwarz inequality, it is easy to see that

$$\epsilon\Big|\int_0^t\int_{\mathbb{S}^1}G_a(R,\theta)\,d\theta\Big|+\epsilon\Big|\int_0^t\int_{\mathbb{S}^1}G_b(R,\theta)\,d\theta\Big| \le C\epsilon\sqrt{t}\Big(\int_0^t||f_t||_{L^2}\Big)^{1/2}.$$

Using (3.341), (3.342) and the previous three inequalities, together with the smallness of M, we conclude that

$$|a_1|+|b_1| \le C(|m_a|+|m_b|)+C||\nabla u||_{L^2(\Omega^\pm)}+C|a_0|+C\sum_{k\ge2}(|a_k|+|b_k|)+C\epsilon\sqrt{t}\Big(\int_0^t||f_t||_{L^2}^2\Big)^{1/2}. \tag{3.343}$$

Note further that the conservation-of-mass law (1.22) implies

$$a_0 = \frac{1}{|\mathbb{S}^1|}\int_{\mathbb{S}^1} f = \frac{m_0}{|\mathbb{S}^1|} - \frac{1}{|\mathbb{S}^1|}\int_\Omega u - \frac{1}{|\mathbb{S}^1|}\int_{\mathbb{S}^1} f^2,$$

where we recall (1.27). From here and the smallness of a_0, we easily infer

$$|a_0| \le C|m_0| + C||u||_{L^2(\Omega^\pm)} + C\sum_{k\ge 1}(a_k^2 + b_k^2). \tag{3.344}$$

Using (3.344) in (3.343), we obtain (3.336). The second claim (3.337) follows from (3.344) and (3.336). □

We use Lemma 3.10 to prove a necessary smallness estimate on $||f||_{H^{2l}}$.

Lemma 3.11 *There exists a constant C such that the following inequality holds:*

$$||f||^2_{H^{2l}} \le C(m_0^2 + m_a^2 + m_b^2) + CM + C\epsilon t M. \tag{3.345}$$

Proof. The boundary condition (3.83) can be rewritten in the following way:

$$-f_{\theta\theta} - f = u\circ\phi - N(f), \tag{3.346}$$

where $N(f)$ is the quadratic nonlinearity defined by (1.9). Apply the differential operator ∂^{2l-1} to both sides of (3.346), multiply with $f_{\theta^{2l-1}}$ and integrate over $\mathbb{S}^1$ to obtain

$$\int_{\mathbb{S}^1}(f_{\theta^{2l}})^2 - (f_{\theta^{2l-1}})^2 = \int_{\mathbb{S}^1}\partial_\theta^{2l-1}\big(u\circ\phi^j\big) f_{\theta^{2l-1}} + \int_{\mathbb{S}^1}\partial_\theta^{2l-1}(N(f)) f_{\theta^{2l-1}} \tag{3.347}$$

From (3.347), using the Leibniz rule, the Cauchy-Schwarz inequality, the trace inequality and the smallness of $||f||_{H^{2l}}$, we can conclude (see (1.20))

$$||\mathbb{P}_{2+}f_{\theta^{2l}}||^2_{L^2} \le C||u||^2_{H^{2l-1}(\Omega^\pm)} + C||\mathbb{P}_1 f||^2_{L^2}.$$

By the Sobolev inequality, we conclude

$$||\mathbb{P}_{2+}f||^2_{H^{2l}} \le C\mathfrak{E} + C||\mathbb{P}_1 f||^2_{L^2} \le C\mathfrak{E} + C\mathcal{E} + C(m_0^2 + m_a^2 + m_b^2) + CM||\mathbb{P}_{2+}f||^2_{L^2} + C\epsilon t\int_0^t ||f_t||^2_{L^2},$$

where we used the estimate (3.336) in the second inequality. This estimate implies

$$||\mathbb{P}_{2+}f||^2_{H^{2l}} \le C(m_0^2 + m_a^2 + m_b^2) + C(\mathfrak{E} + \mathcal{E}) + C\epsilon t\int_0^t ||f_t||^2_{L^2}. \tag{3.348}$$

From (3.348), (3.336) and (3.337), we easily obtain

$$\begin{aligned} ||f||^2_{H^{2l}} &\le C(m_0^2 + m_a^2 + m_b^2) + C(\mathfrak{E} + \mathcal{E}) \le C(m_0^2 + m_a^2 + m_b^2) + C(M/(\alpha\nu^{l-1}) + M) + C\epsilon t M \\ &\le C(m_0^2 + m_a^2 + m_b^2) + CM + C\epsilon t M. \end{aligned}$$

as claimed in the lemma. □

Estimates for $\mathcal{E}_{(0)}$ revisited. Let $M \le L/2$, where L is given by Theorem 3.4 and let (u,f) be the unique solution of the regularized Stefan problem with the initial conditions satisfying $E_\alpha(u_0,f_0) \le M$. Now recall the definition (1.29) and the expansion (1.9) $|g| - 1 = f + f_\theta^2/2 + \Psi(f)$, where $\Psi(f) = O(|f|^3 + |f_\theta|^3)$. We have

$$\begin{aligned} \mathcal{E}_{(0)}(u,f) &= \frac{1}{2}\int_\Omega u^2 + \int_\Omega u + \int_{\mathbb{S}^1}(|g| - 1) = \frac{1}{2}\int_\Omega u^2 + \int_\Omega u + \int_{\mathbb{S}^1} f + \int_{\mathbb{S}^1} f_\theta^2 - \int_{\mathbb{S}^1}\Psi(f) \\ &= \frac{1}{2}\int_\Omega u^2 - \frac{1}{2}\int_{\mathbb{S}^1} f^2 + \int_{\mathbb{S}^1} f_\theta^2 - \int_{\mathbb{S}^1}\Psi(f) = \frac{1}{2}\int_\Omega (\mathbf{P}u)^2\,dx - \frac{1}{2}\int_{\mathbb{S}^1}(\mathbb{P}f)^2 + \big(\frac{1}{2|\Omega|} - \frac{1}{2|\mathbb{S}^1|}\big)\big(\int_\Omega u\big)^2 \\ &\quad + \frac{1}{2|\mathbb{S}^1|}\Big(\big(\int_\Omega u\big)^2 - \big(\int_{\mathbb{S}^1} f\big)^2\Big) + \int_{\mathbb{S}^1} f_\theta^2 - \int_{\mathbb{S}^1}\Psi(f) \\ &= \frac{1}{2}\int_\Omega (\mathbf{P}u)^2\,dx + \frac{D}{2}\big(\int_\Omega u\big)^2 + \frac{1}{2}\int_{\mathbb{S}^1}\{f_\theta^2 - \mathbb{P}f^2\} + \frac{1}{2|\mathbb{S}^1|}\Big(\big(\int_\Omega u\big)^2 - \big(\int_{\mathbb{S}^1} f\big)^2\Big) - \int_{\mathbb{S}^1}\Psi(f). \end{aligned}$$

Note that, again due to (1.22),

$$\Big(\int_\Omega u\Big)^2-\Big(\int_{\mathbb{S}^1} f\Big)^2=(m_0-\int_{\mathbb{S}^1}\frac{f^2}{2})\Big(\int_\Omega u-\int_{\mathbb{S}^1} f\Big), \tag{3.349}$$

and it is hence of third order. Using the estimates (3.336), (3.337) and the equality (3.349), it is easy to see that

$$\Big|\frac{1}{2|\mathbb{S}^1|}\Big(\Big(\int_\Omega u\Big)^2-\Big(\int_{\mathbb{S}^1} f\Big)^2\Big)-\int_{\mathbb{S}^1}\Psi(f)\Big|\le\frac{C}{\lambda}m_0^2+\lambda\Big(\int_\Omega u\Big)^2+\lambda a_0^2$$
$$+C\sqrt{M}\Big(m_0^2+m_a^2+m_b^2+||u||^2_{L^2(\Omega^\pm)}+||\nabla u||^2_{L^2(\Omega)}+\int_{\mathbb{S}^1}\{f_\theta^2-\mathbb{P}f^2\}+\epsilon t\int_0^t||f_t||^2_{L^2}\Big).$$

The last inequality implies that there exists a constant C_* such that

$$\mathcal{E}_{(0)}\ge C_*\big(||u||^2_{L^2(\Omega^\pm)}+\int_{\mathbb{S}^1}\{f_\theta^2-\mathbb{P}f^2\}\big)-C_*(m_0^2+m_a^2+m_b^2)-C_*\sqrt{M}||\nabla u||^2_{L^2(\Omega^\pm)}-C_*\sqrt{M}\epsilon t\int_0^t||f_t||^2_{L^2}. \tag{3.350}$$

Thus, for small M this implies that $\tilde{\mathcal{E}}_{(0)}+C_*(m_0^2+m_a^2+m_b^2)+\mathcal{E}_{(1)}+\int_0^t\mathcal{D}_{(1)}$ is a positive definite quantity. After summing the zero-th and the first level energy identities and adding $C_*(m_0^2+m_a^2+m_b^2)$ to both sides, we obtain

$$C_*(m_0^2+m_a^2+m_b^2)+\mathcal{E}_{(0)}(t)+\hat{\mathcal{E}}^\epsilon_{(0)}(t)+||\nabla u(t)||^2_{L^2(\Omega^\pm)}+\int_0^t||\nabla u||^2_{L^2(\Omega^\pm)}+\int_0^t||u_t||^2_{L^2(\Omega^\pm)}+\int_0^t\int_{\mathbb{S}^1}\{f_{\theta t}^2-f_t^2\}$$
$$=C_*(m_0^2+m_a^2+m_b^2)+\mathcal{E}_{(0)}(0)+\hat{\mathcal{E}}^\epsilon_{(0)}(0)+||\nabla u_0||^2_{L^2(\Omega^\pm)}+\epsilon\int_{\mathbb{S}^1}f_{\theta^2}f_{\theta^4t}\Big(\frac{1}{R|g|}-1\Big)+\epsilon\int_{\mathbb{S}^1}\Big(N(f)+\frac{1}{R}(|g|^{-1})_\theta f_\theta\Big)f_{\theta^4t}$$
$$+\int_0^t\int_\Gamma\Big\{O_1+P_1\Big\}+\int_0^t\int_{\mathbb{S}^1}\Big\{Q_1+S_1\Big\}, \tag{3.351}$$

whereby O_1, P_1, Q_1 and S_1 are defined by dropping the index m in the definitions (3.102) and (3.103). Estimates analogous to (3.149) and (3.150) imply the bound on the ϵ-dependent terms on RHS of (3.351).

$$\Big|\epsilon\int_{\mathbb{S}^1}f_{\theta^2}f_{\theta^4t}\Big(\frac{1}{R|g|}-1\Big)+\epsilon\int_{\mathbb{S}^1}\Big(N(f)+\frac{1}{R}(|g|^{-1})_\theta f_\theta\Big)f_{\theta^4t}\Big|\le C_0\Big(\lambda\int_0^t\mathcal{D}^\epsilon+\lambda\epsilon\int_0^t\mathcal{D}\Big)$$
$$+\frac{C_0t}{\lambda}\sup_{0\le s\le t}\big(||f||^2_{H^3}\hat{\mathcal{E}}^\epsilon_{(0)}\big)+C_0t\epsilon\sup_{0\le s\le t}\big(||f||^2_{H^3}||\mathbb{P}_1f||^2_{L^2}\big). \tag{3.352}$$

3.3.5 Uniform interval of existence: $t^\epsilon\ge T$ for small ϵ

Our objective is to prove that the solution (u^ϵ,f^ϵ) exists on a time interval independent of ϵ.

Lemma 3.12 *There exists a $T>0$ independent of ϵ such that the solution (u^ϵ,f^ϵ) exists on the time interval $[0,T]$ and $E_{\alpha,\nu}$ is continuous on $[0,T[$.*

Proof. As in Subsection 3.3.3, we can pass to the limit as $m\to\infty$ in the estimate (3.284). In doing so, we use the bound (3.345) to ensure the smallness of $||f^m||_{H^l}$. Namely,

$$||f^m||_{H^l}\le||f||_{H^l}+||f^m-f||_{H^l}\le\frac{3}{2}||f||_{H^l}\le C(m_0^2+m_a^2+m_b^2)+CM+C\epsilon tM, \tag{3.353}$$

for m large enough. Note that all the estimates involving the constant $\tilde{C}$ simply vanish, and for any $0\le p\le l-1$, we recover:

$$\sup_{0\le s\le t}\mathfrak{E}_p+\int_0^t\mathfrak{D}_p+(1-C_2t)\Big(\sup_{0\le s\le t}\hat{\mathfrak{E}}^\epsilon_p+\int_0^t\hat{\mathfrak{D}}_p\Big)\le\delta_p+\nu^l\int_0^t\mathfrak{D}_{p+1}+C\sum_{q=0}^{p-1}\int_0^t\mathfrak{D}_q+\epsilon\mathcal{J}_1+C^*\int_0^t\mathcal{D}, \tag{3.354}$$

where $\mathcal{J}_1:=Ct\mathcal{E}^m_{(0)}+C\sup_{0\le s\le t}||f^m||^2_{H^4}\mathcal{E}^m+C||\mathbb{P}_1f^m||^4_{L^2}$. Multiplying (3.354) by ν^p and summing over all $p=0,1,\dots,l-1$, just like in the derivation of (3.286), we can infer

$$\sup_{0\le s\le t}\mathfrak{E}^\nu+\sum_{p=0}^{l-1}d_p\int_0^t\mathfrak{D}_p+(1-C_2t)\Big(\sup_{0\le s\le t}\hat{\mathfrak{E}}^{\epsilon,\nu}+\int_0^t\hat{\mathfrak{D}}^{\epsilon,\nu}\Big)\le\mathfrak{E}^\nu_\epsilon(0)+\epsilon\sum_{q=0}^{l-1}\nu^q\mathcal{J}_1+\sum_{q=0}^{l-1}\nu^qC^*\int_0^t\mathcal{D}, \tag{3.355}$$

whereby we observe that none of the constants in the inequality (3.355) contains ϵ in the denominator. Secondly, we pass to the limit as $m\to\infty$ in the estimate (3.199). Thereby we use the identity (3.351) together with the energy estimate (3.352). Note that all the terms containing the constant $\bar{C}$ vanish and we arrive at:

$$\sup_{0\le s\le t}\mathcal{E}(s)+C_*(m_0^2+m_a^2+m_b^2)+\sup_{0\le s\le t}\hat{\mathcal{E}}^\epsilon(s)+\int_0^t\mathcal{D}_\epsilon\le\mathcal{E}_\epsilon(0)+C_*(m_0^2+m_a^2+m_b^2)+\frac{C_0t}{\lambda}\sup_{0\le s\le t}\hat{\mathcal{E}}^\epsilon_{(0)}(s)$$
$$+C_0t\epsilon\sup_{0\le s\le t}\big(||f||^2_{H^3}||\mathbb{P}_1f||^2_{L^2}\big)+C(\lambda+||f||_{H^4}+\sqrt{M}+M)\int_0^t\mathcal{D}_\epsilon+C(\lambda+||f||_{H^4}+\sqrt{M})\int_0^t\mathfrak{D} \tag{3.356}$$

Multiplying (3.355) by α and summing it with (3.356), we obtain

$$\sup_{0\le s\le t}\mathcal{E}(s)+C_*(m_0^2+m_a^2+m_b^2)+\alpha\sup_{0\le s\le t}\mathfrak{E}^\nu(s)+(1-C(\lambda+||f||_{H^4}+\sqrt{M}+M)-\alpha\sum_{q=0}^{l-1}\nu^qC^*)\int_0^t\mathcal{D}$$
$$+\sum_{p=0}^{l-1}(\alpha d_p-C(\lambda+||f||_{H^4}+\sqrt{M}))\int_0^t\mathfrak{D}_p+(1-C_0t)\sup_{0\le s\le t}\hat{\mathcal{E}}^\epsilon(s)+(1-C(\lambda+||f||_{H^4}+\sqrt{\theta}+\theta))\int_0^t\hat{\mathcal{D}}^\epsilon$$
$$+\alpha(1-C_2t)\big(\sup_{0\le s\le t}\hat{\mathfrak{E}}^{\epsilon,\nu}+\int_0^t\hat{\mathfrak{D}}^{\epsilon,\nu}\big)\le\mathcal{E}_\epsilon(0)+C_*(m_0^2+m_a^2+m_b^2)+\alpha\mathfrak{E}^\nu_\epsilon(0)+\epsilon\mathcal{J}_2, \tag{3.357}$$

where $\mathcal{J}_2:=\alpha\sum_{q=0}^{l-1}\nu^q\mathcal{J}_1+C_0t\sup_{0\le s\le t}\big(||f||^2_{H^3}||\mathbb{P}_1f||^2_{L^2}\big)$. Using the bound (3.353) to ensure the smallness of $||f^m||_{H^l}$ and choosing λ and M sufficiently small, we obtain the following energy bound for any $t\le t^\epsilon$ from (3.357):

$$\sup_{0\le s\le t}\mathcal{E}(s)+C_*(m_0^2+m_a^2+m_b^2)+\alpha\sup_{0\le s\le t}\mathfrak{E}^\nu(s)+\big(\frac{3}{4}-\epsilon CTM\big)\big(\int_0^t\mathcal{D}+\alpha\int_0^t\mathfrak{D}^\nu\big)+(1-C_0t)\sup_{0\le s\le t}\hat{\mathcal{E}}^\epsilon(s)+\frac{3}{4}\int_0^t\hat{\mathcal{D}}^\epsilon$$
$$+\alpha(1-C_2t)\big(\sup_{0\le s\le t}\hat{\mathfrak{E}}^{\epsilon,\nu}+\int_0^t\hat{\mathfrak{D}}^{\epsilon,\nu}\big)\le\mathcal{E}(0)+C_*(m_0^2+m_a^2+m_b^2)+\hat{\mathcal{E}}^\epsilon(0)+\alpha\mathfrak{E}^\nu(0)+\alpha\hat{\mathfrak{E}}^{\epsilon,\nu}(0)+\epsilon\mathcal{J}_2. \tag{3.358}$$

Now extend $[0,t^\epsilon[$ to a maximal interval $[0,\bar{t}^\epsilon[$ such that $E_{\alpha,\nu}(s)+\int_0^sD_{\alpha,\nu}$ is continuous on $[0,\bar{t}^\epsilon[$ and the inequality (3.358) holds for any $0\le t\le\bar{t}^\epsilon$. Let $L\le\theta$, where θ is given by Lemma 3.2. Set

$$\mathcal{T}:=\sup_t\Big\{0<t<\bar{t}^\epsilon:\sup_{0\le s\le t}E_{\alpha,\nu}(u^\epsilon,f^\epsilon)(s)+C_*(m_0^2+m_a^2+m_b^2)+\int_0^tD_{\alpha,\nu}(u^\epsilon,f^\epsilon)(\tau)d\tau\le L\Big\}.$$

Note that $\mathcal{T}\ge t^\epsilon>0$. Furthermore, let us choose $\epsilon_0,\theta>0,T$ small enough such that $C_0T,C_2T<\frac{3}{8}$, and $\epsilon_0CTM<\frac{1}{8}$. Note that $\mathcal{E}_\epsilon(0)+C_*(m_0^2+m_a^2+m_b^2)+\alpha\mathfrak{E}^\nu(0)\le L/2$. From (3.358), we conclude that for any $t\le T$ and any $\epsilon\le\epsilon_0$ the following inequality holds:

$$E_{\alpha,\nu}+C_*(m_0^2+m_a^2+m_b^2)+\int_0^tD_{\alpha,\nu}\le\frac{4}{5}L<L \tag{3.359}$$

However, if $\bar{t}^\epsilon<T$, then $\mathcal{T}<T$ and the inequality (3.359) holds for any $t\le\mathcal{T}$. This contradicts the definition of $\mathcal{T}$ together with the continuity of $\mathcal{E}_\epsilon$ and hence $\bar{t}^\epsilon\ge T$. □

3.4 Proofs of Stability Theorems

Proof of Theorem 1.1.

Since the inequality (3.358) holds on a time interval $[0,T]$, where T does not depend on ϵ, we may pass to the limit as $\epsilon\to0$. The sequence (u^ϵ,f^ϵ) converges to the solution (u,f) of the original Stefan problem. In addition to that $\epsilon CTM\to0$ and $\epsilon\mathcal{J}_2\to0$ as $\epsilon\to0$, since $\mathcal{J}_2$ is bounded. Due to the choice of $(u_0^\epsilon,f_0^\epsilon)$, we have $\hat{\mathcal{E}}^\epsilon(0)$, $\hat{\mathfrak{E}}^{\epsilon,\nu}(0)\to0$, as $\epsilon\to0$. Together with the weak lower semicontinuity we obtain the energy bound:

$$\sup_{0\le s\le t}\Big(\mathcal{E}(s)+C_*(m_0^2+m_a^2+m_b^2)+\alpha\mathfrak{E}^\nu(s)\Big)+\frac{1}{2}\Big(\int_0^t\mathcal{D}(\tau)d\tau+\alpha\int_0^t\mathfrak{D}^\nu(\tau)d\tau\Big)$$
$$\le\mathcal{E}(0)+C_*(m_0^2+m_a^2+m_b^2)+\alpha\mathfrak{E}^\nu(0), \tag{3.360}$$

for any $0 \le t \le T$, where $T = \min(\frac{1}{2C_0}, \frac{1}{2C_2})$. The uniqueness proof is analogous to the proof of uniqueness in Theorem 3.4, where we formally set $\epsilon = 0$. Given the initial data (u_0, f_0) such that the smallness assumptions of Theorem 1.1 are satisfied, we can construct a solution on time interval $[0,T[$. From (3.360), at $t = T/2$, we can conclude $\mathcal{E}(T/2) + (m_a^2 + m_b^2) + \alpha\mathfrak{E}(T/2) \le \mathcal{E}(0) + C_*(m_0^2 + m_a^2 + m_b^2) + \alpha\mathfrak{E}(0)$. The idea is to solve the Stefan problem with new initial data $(u^1(x,0), f^1(\theta,0) = (u(x,\frac{T}{2}), f(\theta,\frac{T}{2}))$. The problem allows for a unique solution that exists at least on a time interval of length $[0,T[$. Due to the uniqueness we have $(u^1(x,t), f^1(\theta,t)) = (u(x,t+\frac{T}{2}), f(\theta,t+\frac{T}{2}))$ and we have thus extended the solution (u,f) to the interval $[0,\frac{3T}{2}[$. Iterating this procedure, we conclude that the solution exists for all $t \ge 0$. Furthermore we conclude that (3.360) holds for any $t > 0$. In order to prove (1.38), we apply the same idea. Let first $M^* \le \frac{M}{4}$ and assume $E_{\alpha,\nu}(u_0,f_0) + C_*(m_0^2 + m_a^2 + m_b^2) \le M^*$. By the above considerations there exists a global solution (u,f) with the initial value (u_0,f_0) and it satisfies the global bound $\sup_{0 \le s \le \infty} E_{\alpha,\nu}(s) + C_*(m_0^2 + m_a^2 + m_b^2) + \int_0^\infty D_{\alpha,\nu}(\tau)\,d\tau \le \frac{M}{2}$. Fix any $s > 0$. We solve the Stefan problem with the new initial data $(u^1(x,0), f^1(\theta,0)) = (u(x,s), f(\theta,s))$. The problem allows for the unique solution since

$$E_{\alpha,\nu}(u_0^1, f_0^1) + C_*(m_0^2 + m_a^2 + m_b^2) = E_{\alpha,\nu}(u,f)(s) + C_*(m_0^2 + m_a^2 + m_b^2) \le \frac{M}{2}.$$

In addition to this we have the global bound $\sup_{0 \le t < \infty} E_{\alpha,\nu}(u^1,f^1)(t) + C_*(m_0^2 + m_a^2 + m_b^2) + \int_0^\infty D_{\alpha,\nu}(u^1,f^1)(\tau)\,d\tau \le M$ (again by the same consideration as above). We are thus in the uniqueness regime and we conclude $(u^1,f^1)(t) = (u,f)(t+s)$ for any $t \ge 0$. We may now use the estimate (3.360) to obtain (1.38).

Proof of Theorem 1.2.

Our first objective is to find the candidate point $(x_0 + \bar{x}_0, \bar{y}_0)$ close to $(x_0, 0)$ and the limiting radius $\bar{R}$, so that the solution $(v(t), \Gamma(t))$ would converge to $(1/\bar{R}, S_{\bar{R}}(x_0 + \bar{x}_0, \bar{y}_0))$ based on the 'mass'- and 'momentum conservation'. To this end, for any pair $(\bar{x}_0, \bar{y}_0)$ define

$$F_a^{\bar{x}_0,\bar{y}_0}(R,\theta) = \int^R \big(r\cos\theta + x_0 + \bar{x}_0 + R_*^2 \frac{r\cos\theta + x_0 + \bar{x}_0}{(r\cos\theta + x_0 + \bar{x}_0)^2 + (r\sin\theta + \bar{y}_0)^2}\big)\,dr,$$

$$F_b^{\bar{x}_0,\bar{y}_0}(R,\theta) = \int^R \big(r\sin\theta + \bar{y}_0 + R_*^2 \frac{r\sin\theta + \bar{y}_0}{(r\cos\theta + x_0 + \bar{x}_0)^2 + (r\sin\theta + \bar{y}_0)^2}\big)\,dr.$$

Choose $\bar{R}$ (close to 1) to be the solution of $\int_\Omega \frac{1}{R} + \frac{1}{2}\int_{\mathbb{S}^1} \bar{R}^2 = \int_\Omega (1 + u_0) + \frac{1}{2}\int_{\mathbb{S}^1} (1 + f_0)^2$.

Lemma 3.13 *There exists a pair $(\bar{x}_0, \bar{y}_0)$ close to $(0,0)$ such that*

$$\int_{\mathbb{S}^1} F_a^{\bar{x}_0,\bar{y}_0}(\bar{R},\theta)\,d\theta = m_a + \int_{\mathbb{S}^1} F_a(1,\theta)\,d\theta; \quad \int_{\mathbb{S}^1} F_b^{\bar{x}_0,\bar{y}_0}(\bar{R},\theta)\,d\theta = m_b + \int_{\mathbb{S}^1} F_b(1,\theta)\,d\theta. \tag{3.361}$$

Proof. The solvability of the above system of equations follows easily from the implicit function theorem. We have to check that at the point $(\bar{x}_0, \bar{y}_0) = (0,0)$ the Jacobian

$$J := \partial_{\bar{x}_0}\Big(\int_{\mathbb{S}^1} F_a^{\bar{x}_0,\bar{y}_0}(\bar{R},\theta)\,d\theta\Big)\partial_{\bar{y}_0}\Big(\int_{\mathbb{S}^1} F_b^{\bar{x}_0,\bar{y}_0}(\bar{R},\theta)\,d\theta\Big) - \partial_{\bar{x}_0}\Big(\int_{\mathbb{S}^1} F_b^{\bar{x}_0,\bar{y}_0}(\bar{R},\theta)\,d\theta\Big)\partial_{\bar{y}_0}\Big(\int_{\mathbb{S}^1} F_a^{\bar{x}_0,\bar{y}_0}(\bar{R},\theta)\,d\theta\Big)$$

is different from zero. To do this, we first easily check that at $(\bar{x}_0, \bar{y}_0) = (0,0)$

$$\partial_{\bar{x}_0} F_a^{\bar{x}_0,\bar{y}_0}(\bar{R},\theta) = \int^{\bar{R}} \big(1 + R_*^2 \frac{1}{r^2 + x_0^2 + 2rx_0\cos\theta} - R_*^2 \frac{2(r\cos\theta + x_0)^2}{(r^2 + x_0^2 + 2rx_0\cos\theta)^2}\big)\,dr = \bar{R} + R_*^2 \frac{x_0\cos\theta + \bar{R}\cos 2\theta}{\bar{R}^2 + x_0^2 + 2x_0\cos\theta}.$$

From here, by a standard computation, we conclude that at $(\bar{x}_0, \bar{y}_0) = (0,0)$

$$\int_{\mathbb{S}^1} \partial_{\bar{x}_0} F_a^{\bar{x}_0,\bar{y}_0}(\bar{R},\theta)\,d\theta = \int_{\mathbb{S}^1} \big(\bar{R} + R_*^2 \frac{x_0\cos\theta + \bar{R}\cos 2\theta}{\bar{R}^2 + x_0^2 + 2x_0\cos\theta}\big)\,d\theta$$
$$= 2\pi\bar{R} + \frac{R_*^2\bar{R}}{2x_0^2}\Big\{\theta + 2\arctan\big(\frac{(\bar{R} + x_0)\cos\frac{\theta}{2}}{\bar{R}\sin\frac{\theta}{2} - x_0\sin\frac{\theta}{2}}\big)\Big\}\Big|_{-\pi}^{\pi} - R_*^2 \frac{\sin\theta}{x_0}\Big|_{-\pi}^{\pi} = 2\pi\bar{R} + 0 = 2\pi\bar{R}.$$

By a similar calculation, we conclude that at $(\bar{x}_0,\bar{y}_0)=(0,0)$

$$\partial_{\bar{y}_0}\Big(\int_{\mathbb{S}^1} F_b^{\bar{x}_0,\bar{y}_0}(1,\theta)\,d\theta\Big)=2\pi\bar{R};\quad \partial_{\bar{x}_0}\Big(\int_{\mathbb{S}^1} F_b^{\bar{x}_0,\bar{y}_0}(1,\theta)\,d\theta\Big)=0;\quad \partial_{\bar{y}_0}\Big(\int_{\mathbb{S}^1} F_a^{\bar{x}_0,\bar{y}_0}(1,\theta)\,d\theta\Big)=0.$$

Hence $J=4\pi^2\bar{R}^2\neq 0$. This finishes the proof of the lemma. □

We parametrize Γ as a 'graph' over $S_{\bar{R}}(x_0+\bar{x}_0,\bar{y}_0)$ - the unit sphere centered at $(x_0+\bar{x}_0,\bar{y}_0)$. We set $\Gamma=(\tilde{R}\cos\tilde{\theta}+x_0+\bar{x}_0,\tilde{R}\sin\tilde{\theta}+\bar{y}_0)$, $\tilde{R}=1+\tilde{f}$ and $v=1/\bar{R}+\tilde{u}$. With respect to the new parametrization of Γ, the conservation laws (1.24) take the form

$$\partial_t\int_\Omega \tilde{u}+\partial_t\int_{\mathbb{S}^1}\{\bar{R}\tilde{f}+\frac{\tilde{f}^2}{2}\};\quad \partial_t\int_\Omega up_a+\partial_t\int_{\mathbb{S}^1}F_a^{\bar{x}_0,\bar{y}_0}(\tilde{R},\tilde{\theta})=0;\quad \partial_t\int_\Omega up_b+\partial_t\int_{\mathbb{S}^1}F_b^{\bar{x}_0,\bar{y}_0}(\tilde{R},\tilde{\theta})=0. \tag{3.362}$$

After integrating the above conservation laws in time, we note that our choice of the center $(x_0+\bar{x}_0,\bar{y}_0)$ and the radius $\bar{R}$ of the shifted circle, together with the definitions (1.27) and (1.28) of m_a and m_b, implies

$$\begin{gathered}\int_\Omega up_a+\int_{\mathbb{S}^1}\{F_a^{\bar{x}_0,\bar{y}_0}(\tilde{R},\tilde{\theta})-F_a^{\bar{x}_0,\bar{y}_0}(1,\tilde{\theta})\,d\tilde{\theta}\}=0;\quad \int_\Omega up_b+\int_{\mathbb{S}^1}\{F_b^{\bar{x}_0,\bar{y}_0}(\tilde{R},\tilde{\theta})-F_b^{\bar{x}_0,\bar{y}_0}(1,\tilde{\theta})\,d\tilde{\theta}\}=0;\\ \int_\Omega \tilde{u}+\int_{\mathbb{S}^1}\{\bar{R}\tilde{f}+\frac{\tilde{f}^2}{2}\}=0.\end{gathered} \tag{3.363}$$

The coefficients in the Fourier expansion of $\tilde{f}$ will be denoted by $\tilde{a}_k$ and $\tilde{b}_k$, i.e. $\tilde{f}=\sum_{k=0}^\infty \tilde{a}_k\cos k\tilde{\theta}+\tilde{b}_k\sin k\tilde{\theta}$. We also introduce the map $\tilde{\phi}:[0,\infty[\times\mathbb{S}^1\to[0,\infty[\times\Gamma$, $\tilde{\phi}(t,\theta):=(\tilde{R}(t,\tilde{\theta})\cos\tilde{\theta}+x_0+\bar{x}_0,\tilde{R}(t,\tilde{\theta})\sin\tilde{\theta}+\bar{y}_0)$. With respect to the parametrization $\tilde{\phi}$, we redefine the zero-th order energy $\mathcal{E}_{(0)}$:

$$\mathcal{E}_{(0)}(\tilde{u},\tilde{f})=\frac{1}{2}\int_\Omega\{(1/\bar{R}+\tilde{u})^2-1/\bar{R}\}+\int_{\mathbb{S}^1}\{\sqrt{(\bar{R}+\tilde{f})^2+\tilde{f}_\theta^2}-\bar{R}\}.$$

The only difference from the definition (1.29) is in subtracting $1/\bar{R}$ instead of 1 in the first integral, and $\bar{R}$ instead of 1 in the second integral above. The higher order energies $\mathcal{E}_{(+)}$ and $\mathcal{D}_{(+)}$ are defined precisely as before.

Lemma 3.14 *The temporal energy $\mathcal{E}(\tilde{u},\tilde{f})$ is positive definite.*

Proof. Using the new conservation laws (3.363), by a small perturbation of (3.338) and (3.339), as in Lemma 3.10, we can bound the first and zero-th order momenta of $\tilde{f}$:

$$|\tilde{a}_1|+|\tilde{b}_1|\le C\Big(||\nabla\tilde{u}||_{L^2(\Omega^\pm)}+C\sum_{k\ge 2}(|\tilde{a}_k|+|\tilde{b}_k|)\Big),\quad |\tilde{a}_0|\le C\Big(||\tilde{u}||_{L^2(\Omega^\pm)}+||\nabla\tilde{u}||_{L^2(\Omega^\pm)}+\sum_{k\ge 2}(|\tilde{a}_k|+|\tilde{b}_k|)\Big). \tag{3.364}$$

With the above bounds we obtain an estimate analogous to (3.350):

$$\mathcal{E}_{(0)}(\tilde{u},\tilde{f})\ge C_1\big(||\tilde{u}||^2_{L^2(\Omega^\pm)}+\int_{\mathbb{S}^1}\{\tilde{f}_\theta^2-\mathbb{P}\tilde{f}^2\}\big)-C_1\sqrt{M}||\nabla\tilde{u}||^2_{L^2(\Omega^\pm)},$$

where from we easily infer the claim of the lemma. □

The following lemma states the analogue of the stability estimate (1.38), where the energy quantities are expressed as functions of $\tilde{u}$ and $\tilde{f}$.

Lemma 3.15 *The following inequality holds:*

$$E_{\alpha,\nu}(\tilde{u},\tilde{f})(t)+\frac{1}{2}\int_s^t D_{\alpha,\nu}(\tilde{u},\tilde{f})(\tau)\,d\tau\le E_{\alpha,\nu}(\tilde{u},\tilde{f})(s),\quad t\ge s\ge 0.$$

Proof. The proof of this lemma is completely analogous to the proof of (1.38). Formally, the only difference is the absence of the term $C_*(m_0^2+m_a^2+m_b^2)$ in the statement of the inequality. The reason for this is Lemma 3.14, which (again, due to our definition of $\tilde{u}$ and $\tilde{f}$) enables us to conclude positivity of $\mathcal{E}(\tilde{u},\tilde{f})$ without adding any constant to it. □

From now on, we shall abuse the notation and denote $\tilde{u}$, $\tilde{R}$, $\tilde{f}$, $\tilde{\theta}$ and $\tilde{\phi}$ respectively by u, R, f, θ and ϕ. The main ingredient in proving the exponential decay is to control the instant energy in terms of the dissipation, as stated in the following lemma:

Lemma 3.16 *There exists a constant $C>0$ such that $E_{\alpha,\nu}\le CD_{\alpha,\nu}$.*

Proof. Note that the only term, which a-priori can not be bounded by a constant multiple of D, is precisely $\mathcal{E}_{(0)}$. Recall the expansion (1.9): $u\circ\phi=-f-f_{\theta\theta}+N(f)$. For any $k\ge 2$, let us multiply both sides of the first relation in (1.9) by $\cos k\theta$ and $\sin k\theta$ respectively, and integrate over $\mathbb{S}^1$. Observing that $\int_{\mathbb{S}^1}(-f-f_{\theta\theta})\cos k\theta=\frac{(k^2-1)|\mathbb{S}^1|}{2}a_k$, we get

$$a_k=\frac{2}{(k^2-1)|\mathbb{S}^1|}\int_{\mathbb{S}^1}u\circ\phi\cos k\theta-\frac{2}{(k^2-1)|\mathbb{S}^1|}\int_{\mathbb{S}^1}N(f)\cos k\theta \tag{3.365}$$

Note that

$$\Big|\int_{\mathbb{S}^1}u\circ\phi\cos k\theta\Big|=\Big|\int_{\mathbb{S}^1}\big(u\circ\phi-\frac{1}{|\mathbb{S}^1|}\int_{\mathbb{S}^1}u\circ\phi\big)\cos k\theta\Big|\le C||u\circ\phi-\frac{1}{|\mathbb{S}^1|}\int_{\mathbb{S}^1}u\circ\phi||_{L^2}\le C||\nabla u||_{L^2(\Omega^\pm)},$$

where we use the Sobolev inequality in the last estimate above. The last inequality, together with (3.365), immediately implies

$$|a_k|\le C\sqrt{\mathcal{D}}+C\Big|\int_{\mathbb{S}^1}N(f)\cos k\theta\Big|,\ k\ge 2 \tag{3.366}$$

and fully analogously, we arrive at

$$|b_k|\le C\sqrt{\mathcal{D}}+C\Big|\int_{\mathbb{S}^1}N(f)\sin k\theta\Big|,\ k\ge 2. \tag{3.367}$$

In order to estimate a_0, we first invoke the decomposition

$$u\circ\phi|g|=-f-f_{\theta\theta}+q(f), \tag{3.368}$$

where $q(f)$ stands for the nonlinear remainder with the leading quadratic order. Assuming without loss of generality that $\bar{R}=1$ in the last relation in (3.363), we integrating (3.368) over $\mathbb{S}^1$, we find

$$\int_\Gamma u=-\int_{\mathbb{S}^1}f+\int_{\mathbb{S}^1}q(f). \tag{3.369}$$

Multiplying the conservation law $\int_\Omega u+\int_{\mathbb{S}^1}\{f+f^2/2\}=0$ by $\frac{1}{|\Omega|}$ and (3.369) by $\frac{1}{|\Gamma|}$ and subtracting the two equations, we obtain

$$\big(\frac{1}{|\Omega|}-\frac{1}{|\Gamma|}\big)\int_{\mathbb{S}^1}f=-\big(\frac{1}{|\Omega|}\int_\Omega u-\frac{1}{|\Gamma|}\int_\Gamma u\big)-\frac{1}{2|\Omega|}\int_{\mathbb{S}^1}f^2+\frac{1}{|\Gamma|}\int_{|\mathbb{S}^1|}q(f). \tag{3.370}$$

Note that $\zeta-\big(\frac{1}{|\Omega|}-\frac{1}{|\Gamma|}\big)=\frac{1}{|\Gamma|}-\frac{1}{|\mathbb{S}^1|}$ and hence $|\zeta-\big(\frac{1}{|\Omega|}-\frac{1}{|\Gamma|}\big)|\le C(M^*)^{1/2}$, which, for M^* small enough implies that $|\frac{1}{|\Omega|}-\frac{1}{|\Gamma|}|\ge\zeta/2>0$. Hence, upon dividing (3.370) by $K_1:=\frac{1}{|\mathbb{S}^1|}\big(\frac{1}{|\Omega|}-\frac{1}{|\Gamma|}\big)$, we conclude

$$a_0=-\frac{1}{K_1}\Big(\frac{1}{|\Omega|}\int_\Omega u-\frac{1}{|\Gamma|}\int_\Gamma u\Big)-\frac{1}{2K_1|\Omega|}\int_{\mathbb{S}^1}f^2+\frac{1}{K_1|\Gamma|}\int_{\mathbb{S}^1}q(f). \tag{3.371}$$

From the mean value theorem, we deduce

$$\Big|\frac{1}{|\Omega|}\int_\Omega u-\frac{1}{|\Gamma|}\int_\Gamma u\Big|\le C||\nabla u||_{L^2(\Omega^\pm)}.$$

Thus, from (3.371) and the previous inequality, we deduce

$$|a_0|\le C||\nabla u||_{L^2(\Omega^\pm)}+C||f||^2_{L^2} \tag{3.372}$$

Summing (3.372), (3.364), (3.366) and (3.367), we obtain

$$\begin{aligned}|a_0|+\sum_{k=1}^\infty\big(|a_k|+|b_k|\big)&\le C||\nabla u||_{L^2(\Omega^\pm)}+C||f||^2_{L^2}+C\sum_{k=2}^\infty\Big\{\Big|\int_{\mathbb{S}^1}N(f)\cos k\theta\Big|+C\Big|\int_{\mathbb{S}^1}N(f)\sin k\theta\Big|\Big\}\\&\le C||\nabla u||_{L^2(\Omega^\pm)}+C||f||^2_{L^2}+C||N(f)||_{L^2}\le C||\nabla u||_{L^2(\Omega^\pm)}+C||f||^2_{L^2}.\end{aligned} \tag{3.373}$$

Using the smallness of $||f||_{L^2}$, from (3.373) we obtain

$$|a_0|+\sum_{k=1}^{\infty}\big(|a_k|+|b_k|\big)\leq C||\nabla u||_{L^2(\Omega^\pm)}\leq C\sqrt{\mathcal{D}}. \tag{3.374}$$

Finally, from the previous inequality and the conservation law (1.22), we immediately deduce

$$\Big|\int_\Omega u\Big|\leq C\sqrt{\mathcal{D}}. \tag{3.375}$$

The inequalities (3.374) and (3.375) imply the bound $\mathcal{E}_{(0)}\leq C\mathcal{D}$, which in turn implies the estimate claimed in the lemma. □

Using the same argument as in [21] to prove decay, we use the inequality (1.38) and Lemma 3.16 to conclude that there exist constants $k_1,k_2>0$ such that

$$E_{\alpha,\nu}(u,f)(t)\leq k_1 e^{-k_2 t},\ t\geq 0.$$

This finishes the proof of Theorem 1.2.

4 The unstable case $\zeta<0$

4.1 A-priori estimates

In order to bound the higher-order space-time energy, we first introduce the necessary notations. We denote $\mathfrak{M}_q^j:=\mathfrak{M}_q(u^j,f^j)$, $\hat{\mathfrak{M}}_q^j:=\hat{\mathfrak{M}}_q^\epsilon(u^j,f^j)$, for any $j\in\mathbb{N}$, whereby in the definition (2.53), Ω is substituted by Ω^j. Analogously we define $\mathfrak{N}_q^j$ and $\hat{\mathfrak{N}}_q^j$. For any $j\in\mathbb{N}$ we finally set $\mathfrak{M}^j=\sum_{q=1}^k\mathfrak{M}_q^j$ and we define $\hat{\mathfrak{M}}^j$ analogously. Summing the ϵ-independent and the ϵ-dependent energies $\mathfrak{M}^j$ and $\hat{\mathfrak{M}}^j$, we introduce $\mathfrak{M}_\epsilon^j:=\mathfrak{M}^j+\hat{\mathfrak{M}}^j$. In a fully analogous fashion, we introduce $\mathfrak{N}^j$, $\hat{\mathfrak{N}}^j$ and $\mathfrak{N}_\epsilon^j$. Let us introduce

$$M_\beta^j=\mathcal{M}_\epsilon^j+\beta\mathfrak{M}_\epsilon^j;\quad N_\beta^j=\mathcal{N}_\epsilon^j+\beta\mathfrak{N}_\epsilon^j.$$

In analogy to Lemma 3.2, we state the a-priori estimates for the unstable case $\zeta<0$.

Lemma 4.1 *Let $\zeta<0$. Then there exist positive constants β, C_B, ξ_0 and τ^ϵ, such that for any $\xi,\xi_1<\xi_0$, $t\leq\tau^\epsilon$ such that if $M_\beta(0)<\xi/2$ and*

$$M_\beta^m(t)+\int_0^t N_\beta^m(\tau)\,d\tau\leq\xi,\quad \sup_{0\leq s\leq t}||f^m||_{L^2}^2\leq C_B(\xi_1+\int_0^t\mathfrak{N}^m)$$

then

$$M_\beta^{m+1}(t)+\int_0^t N_\beta^{m+1}(\tau)\,d\tau\leq\xi\quad \textit{and}\quad \sup_{0\leq s\leq t}||f^{m+1}||_{L^2}^2\leq C_B(\xi_1+t\int_0^t\mathfrak{N}^{m+1}). \tag{4.376}$$

Remark. Note that the norm $||f_\theta||_{H^{2l-1}}$ is already contained in the definitions of $\mathcal{M}$ and $\mathfrak{M}$ and hence, at variance with the formulation of Lemma 3.2, the second claim of Lemma 4.1 is only concerned with the L^2-norm of f^{m+1}.

Remark. We claim that for any $0\leq k\leq l-1$ the following inequality holds:

$$\int_0^t\Big(||f^{m+1}_{t^{k+1}}||_{L^2}^2+2\epsilon||f^{m+1}_{\theta^2t^{k+1}}||_{L^2}^2+\epsilon^2||f^{m+1}_{\theta^4t^{k+1}}||_{L^2}^2\Big)\leq\frac{C}{\eta}\int_0^t||u^{m+1}||_{L^2(\Omega^m)}^2+\eta\int_0^t\mathfrak{N}^{m+1}_{l-1-k}. \tag{4.377}$$

This claim follows by first applying the differential operator ∂_{t^k} to the regularized jump equation (3.85), taking the squares and integrating over $\mathbb{S}^1$. By the standard trace inequality and the assumptions of smallness on $\mathcal{M}_\epsilon^m$ and $\mathfrak{M}_\epsilon^m$, it is straightforward to get

$$||\partial_{t^k}([u_n^{m+1}]_-^+\circ\phi^m)||_{L^2}\leq\eta\sum_{q=0}^k||\nabla^2u_{t^q}||_{L^2(\Omega^m)}^2+\frac{C}{\eta}\sum_{q=0}^k||\nabla u_{t^q}^{m+1}||_{L^2(\Omega^m)}^2.$$

We now exploit the fact that $u_t^{m+1}=\Delta u^{m+1}$ to conclude that $\sum_{q=0}^k ||\nabla u_{t^q}^{m+1}||^2_{L^2(\Omega^m)} \leq C||\nabla u^{m+1}||^2_{H^{2k}(\Omega^m)}$. Using the standard interpolation inequality and the definition of $\mathfrak{N}^j$, we obtain (4.377).

Proof of Lemma 4.1. The second claim of (4.376) is proved in the same way as the inequality (3.136) is deduced from the jump relation (3.133). The proof of the first claim of the lemma is analogous to the proof of the first claim of (3.115) in Lemma 3.2. The energy identities from Section 2 are nearly the same, except for the additional quadratic terms that appear in the definitions (2.75) and (2.76) of Q^u and S^u, respectively. We first analyze the temporal energies: the identity (2.72) implies

$$\mathcal{M}_\epsilon(s)+\int_0^t \mathcal{N}_\epsilon(s)\,ds=\mathcal{M}_\epsilon(0)+\int_0^t\int_{\Gamma^m}\big\{O^{m,u}+P^{m,u}\big\}+\int_0^t\int_{\mathbb{S}^1}\big\{Q^{m,u}+S^{m,u}\big\}. \tag{4.378}$$

Here $O^{m,u}:=\sum_{k=0}^{l-1}O_k^{m,u}$, $P^{m,u}=\sum_{k=0}^{l-1}P_k^{m,u}$, $Q^{m,u}=\sum_{k=0}^{l-1}Q_k^{m,u}$ and $S^{m,u}=\sum_{k=0}^{l-1}S_k^{m,u}$, whereby

$$O_k^{m,u}:=O^u(\partial_{t^k}u^{m+1}),\qquad P_k^{m,u}:=P^u(\partial_{t^k}u^{m+1}) \tag{4.379}$$

and

$$Q_k^{m,u}:=Q^u(R_{t^k}^m,R_{t^k}^{m+1},R^m),\qquad S_k^{m,u}:=S^u(R_{t^k}^m,R_{t^k}^{m+1},R^m). \tag{4.380}$$

Estimating RHS of (4.378) is analogous to the estimates from Subsection 3.2.3. Namley, $O^{m,u}=O^m$, $P^{m,u}=P^m$ and

$$\begin{aligned}Q^{m,u}&=Q^m+\sum_{k=0}^{l-1}\int_0^t\int_{\mathbb{S}^1}\big\{(\mathbb{P}_0+\mathbb{P}_1)f_{t^k}^{m+1}(\mathbb{P}_0+\mathbb{P}_1)f_{t^{k+1}}^{m+1}+\epsilon\mathbb{P}_1f_{\theta^2t^k}^{m+1}\mathbb{P}_1f_{\theta^2t^{k+1}}^{m+1}\big\};\\ S^{m,u}&=S^m+\sum_{k=0}^{l-1}\int_0^t\int_{\mathbb{S}^1}\big\{((\mathbb{P}_0+\mathbb{P}_1)f_{t^{k+1}}^{m+1})^2+\epsilon(\mathbb{P}_1f_{\theta^2t^{k+1}}^{m+1})^2\big\}.\end{aligned} \tag{4.381}$$

We estimate the second term on RHS of the first equation in (4.381) in the following manner:

$$\begin{aligned}&\Big|\sum_{k=0}^{l-1}\int_0^t\int_{\mathbb{S}^1}\big\{(\mathbb{P}_0+\mathbb{P}_1)f_{t^k}^{m+1}(\mathbb{P}_0+\mathbb{P}_1)f_{t^{k+1}}^{m+1}+\epsilon\mathbb{P}_1f_{\theta^2t^k}^{m+1}\mathbb{P}_1f_{\theta^2t^{k+1}}^{m+1}\big\}\Big|\leq\sum_{k=0}^{l-1}\Big\{\lambda\Big\{\int_0^t||f_{t^{k+1}}^{m+1}||^2_{L^2}+\epsilon||f_{\theta^2t^{k+1}}^{m+1}||^2_{L^2}\Big\}\\&+\frac{C}{\lambda}\int_0^t\big\{||f_{t^k}^{m+1}||^2_{L^2}+\epsilon||f_{\theta^2t^k}^{m+1}||^2_{L^2}\big\}\Big\}\leq\lambda\int_0^t\mathcal{N}_\epsilon^{m+1}+\eta\int_0^t\mathfrak{N}^{m+1}+\frac{C}{\eta}\int_0^t||u^{m+1}||^2_{L^2(\Omega^m)}\\&+\frac{CC_B}{\lambda}(\xi_1+t\int_0^t\mathfrak{N}^{m+1})+\frac{C}{\lambda}\int_0^t\hat{\mathcal{M}}^{m+1},\end{aligned} \tag{4.382}$$

where, in the second inequality we used the estimate (4.377) and the second inequality in (4.376) to bound $\int_0^t\big\{||f^{m+1}||^2_{L^2}+\epsilon||f_{\theta^2}^{m+1}||^2_{L^2}\big\}$ by the last two terms on the right-most side of (4.382). The second term on RHS of the second equation in (4.381) is again estimated using the estimate (4.377):

$$\Big|\sum_{k=0}^{l-1}\int_0^t\int_{\mathbb{S}^1}\big\{(\mathbb{P}_1f_{t^{k+1}}^{m+1})^2+\epsilon(\mathbb{P}_1f_{\theta^2t^{k+1}}^{m+1})^2\big\}\Big|\leq\eta\int_0^t\mathfrak{N}_\epsilon^{m+1}+\frac{C}{\eta}\int_0^t||u^{m+1}||^2_{L^2(\Omega^m)}. \tag{4.383}$$

Using the identity (4.378) and the estimates analogous to (3.144) - (3.150), (3.155), (3.162), (3.164) (3.170), (3.171) - (3.177), (3.181), (3.190) and (3.191) - (3.176), together with the additional estimates (4.382) and (4.383), we arrive at the following inequality:

$$\begin{aligned}&\mathcal{M}_\epsilon^{m+1}+\int_0^t\mathcal{N}_\epsilon^{m+1}\leq\mathcal{M}_\epsilon(0)+\frac{\tilde{C}t}{\lambda}\big(\mathcal{M}^m+\mathcal{M}^{m+1}\big)+\big(\frac{\bar{C}t}{\eta\lambda}+\frac{\bar{C}t}{\epsilon\eta\lambda}\big)\mathcal{M}^{m+1}\\&+C\big(\lambda+\sup_{0\leq s\leq t}||f^m||_{H^4}+\sqrt{\mathfrak{M}^m}\big)\big(\int_0^t\mathcal{N}_\epsilon^m+\int_0^t\mathcal{N}_\epsilon^{m+1}\big)+\frac{CC_B}{\lambda}(\xi_1+t\int_0^t\mathfrak{N}^{m+1})+\frac{C}{\lambda}\int_0^t\hat{\mathcal{M}}^{m+1}\\&+\big(C\lambda+\frac{\bar{C}\eta}{\lambda}+\frac{\bar{C}\eta}{\epsilon\lambda}+C\sqrt{\mathcal{M}^m}\big)\int_0^t\mathfrak{N}^{m+1}+C\int_0^t\mathcal{N}^m\sup_{0\leq s\leq t}\mathfrak{M}^{m+1}(s)+\frac{C}{\lambda}\int_0^t||u^{m+1}||^2_{L^2(\Omega^m)}.\end{aligned} \tag{4.384}$$

We now turn our attention to the estimates for the space-time energies $\mathfrak{M}^j$ and $\mathfrak{N}^j$. Note that the estimates (3.232), (3.238), (3.241) carry over to the unstable case, where the energy terms $\mathfrak{D}^m$, $\mathfrak{D}^{m+1}$, $\mathcal{D}^{m+1}$

are replaced by $\mathfrak{N}^m$, $\mathfrak{N}^{m+1}$ and $\mathcal{N}^{m+1}$, respectively. The identity (3.251) reads

$$\begin{aligned}&\int_0^t\int_{\Gamma^m}\partial_{s^{2p+1}}\partial_{t^c}^*u^{m+1}\partial_{s^{2p+1}}\partial_{t^c}^*(V^{m+1}+\epsilon\Lambda^{m+1})+\frac{1}{2}\int_{\mathbb{S}^1}\left\{|f^{m+1}_{\theta^{2p+2}t^c}|^2-|\mathbb{P}_{2+}f^{m+1}_{\theta^{2p+1}t^c}|^2\right\}\Big|_0^t\\&+\frac{\epsilon}{2}\int_{\mathbb{S}^1}\left\{|f^{m+1}_{\theta^{2p+4}t^c}|^2-|\mathbb{P}_{2+}f^{m+1}_{\theta^{2p+3}t^c}|^2\right\}\Big|_0^t=\int_0^t\int_{\mathbb{S}^1}Q^u(f^m_{\theta^{2p+1}t^c},f^{m+1}_{\theta^{2p+1}t^c},R^m,|g^m|^{2p+1}),\end{aligned}\tag{4.385}$$

where Q^u is defined by (2.75). For the sake of simplicity, let us denote $\tilde{Q}^{m,u}_{2p+1,c}:=Q^u(f^m_{\theta^{2p+1}t^c},f^{m+1}_{\theta^{2p+1}t^c},R^m,|g^m|^{2p+1})$ and note that

$$\tilde{Q}^{m,u}_{2p+1,c}=\tilde{Q}^m_{2p+1,c}+\int_0^t\int_{\mathbb{S}^1}\mathbb{P}_1f^{m+1}_{\theta^{2p+2}t^c}\mathbb{P}_1f^{m+1}_{\theta^{2p+2}t^{c+1}}+\epsilon\int_0^t\int_{\mathbb{S}^1}\mathbb{P}_1f^{m+1}_{\theta^{2p+4}t^c}\mathbb{P}_1f^{m+1}_{\theta^{2p+4}t^{c+1}}.$$

The estimates analogous to (3.254), (3.255), (3.256), (3.263) - (3.264) imply the bound on $\tilde{Q}^m_{2p+1,c}$:

$$\left|\tilde{Q}^m_{2p+1,c}\right|\le C\big(t+\sqrt{\mathfrak{M}^m}+\lambda+\sup_{0\le s\le t}||f^m||_{L^2}\big)\Big(\int_0^t\mathfrak{N}^m_p+\int_0^t\mathfrak{N}^{m+1}_p\Big)+\big(\frac{\tilde{C}t}{\lambda}+\frac{\tilde{C}t}{\lambda\epsilon^4}\big)\sup_{0\le s\le t}\big(\mathfrak{M}^m_p+\mathfrak{M}^{m+1}_p\big).\tag{4.386}$$

Furthermore,

$$\begin{aligned}&\Big|\int_0^t\int_{\mathbb{S}^1}\mathbb{P}_1f^{m+1}_{\theta^{2p+2}t^c}\mathbb{P}_1f^{m+1}_{\theta^{2p+2}t^{c+1}}+\epsilon\int_0^t\int_{\mathbb{S}^1}\mathbb{P}_1f^{m+1}_{\theta^{2p+4}t^c}\mathbb{P}_1f^{m+1}_{\theta^{2p+4}t^{c+1}}\Big|\le C\Big|\int_0^t\int_{\mathbb{S}^1}\mathbb{P}_1f^{m+1}_{t^c}\mathbb{P}_1f^{m+1}_{t^{c+1}}\Big|\\&\le\lambda\int_0^t||\mathbb{P}_1f^{m+1}_{t^{c+1}}||^2_{L^2}+\frac{C}{\lambda}\int_0^t||\mathbb{P}_1f^{m+1}_{t^c}||^2_{L^2}\le\lambda\int_0^t\mathcal{N}^{m+1}+\frac{C}{\lambda}\int_0^t\big(\eta\mathfrak{N}^{m+1}_p+\frac{C}{\eta}||u^{m+1}||^2_{L^2(\Omega^m)}\big)+\frac{C}{\lambda}\int_0^t||f^{m+1}||^2_{L^2},\end{aligned}\tag{4.387}$$

where we used the estimate (4.377) in the third inequality above. Observe that the identity (3.278) reads

$$\begin{aligned}&\int_0^t\int_{\Gamma^m}\partial_{s^{2p}}\partial^*_{t^{c+1}}u^{m+1}\partial_{s^{2p}}\partial^*_{t^c}[u^{m+1}_n]^+_-=\int_0^t\int_{\mathbb{S}^1}\Big\{|f^{m+1}_{\theta^{2p+1}t^{c+1}}|^2-|\mathbb{P}_{2+}f^{m+1}_{\theta^{2p}t^{c+1}}|^2\Big\}\\&+\epsilon\int_0^t\int_{\mathbb{S}^1}\Big\{|f^{m+1}_{\theta^{2p+3}t^{c+1}}|^2-|\mathbb{P}_{2+}f^{m+1}_{\theta^{2p+2}t^{c+1}}|^2\Big\}=\int_0^t\int_{\mathbb{S}^1}S^u(f^m_{\theta^{2p}t^c},f^{m+1}_{\theta^{2p}t^c},R^m,|g^m|^{2p}),\end{aligned}\tag{4.388}$$

where S^u is defined by (2.76). For the sake of simplicity, let us denote $\tilde{S}^{m,u}_{2p,c}:=S^u(f^m_{\theta^{2p}t^c},f^{m+1}_{\theta^{2p}t^c},R^m,|g^m|^{2p+1})$ and note that

$$\tilde{S}^{m,u}_{2p,c}=\tilde{S}^m_{2p,c}+\int_0^t\int_{\mathbb{S}^1}|\mathbb{P}_1f^{m+1}_{\theta^{2p+1}t^{c+1}}|^2+\epsilon\int_0^t\int_{\mathbb{S}^1}|\mathbb{P}_1f^{m+1}_{\theta^{2p+3}t^{c+1}}|^2$$

The estimates analogous to (3.279) - (3.283) imply the bound on $\tilde{S}^m_{2p,c}$:

$$|\tilde{S}^m_{2p,c}|\le C\big(\sqrt{\mathfrak{M}^m}+\lambda+\sup_{0\le s\le t}||f^m||_{L^2}\big)\Big(\int_0^t\big(\mathfrak{N}^m_p+\hat{\mathfrak{N}}^m_p\big)+\int_0^t\big(\mathfrak{N}^{m+1}_p+\hat{\mathfrak{N}}^{m+1}_p\big)\Big)+\frac{\tilde{C}\eta}{\lambda}\int_0^t\mathfrak{N}^{m+1}_p+\frac{\tilde{C}t}{\eta\lambda\epsilon}\mathfrak{M}^{m+1}_p.\tag{4.389}$$

Furthermore, analogously to (4.387), using the estimate (4.377) we obtain

$$\Big|\int_0^t\int_{\mathbb{S}^1}|\mathbb{P}_1f^{m+1}_{\theta^{2p+1}t^{c+1}}|^2+\epsilon\int_0^t\int_{\mathbb{S}^1}|\mathbb{P}_1f^{m+1}_{\theta^{2p+3}t^{c+1}}|^2\Big|\le\lambda\int_0^t\mathcal{N}^{m+1}+\frac{C\eta}{\lambda}\int_0^t\mathfrak{N}^{m+1}_p+\frac{C}{\eta\lambda}\int_0^t||u^{m+1}||^2_{L^2(\Omega^m)}.\tag{4.390}$$

From the argument in the paragraph preceding (4.385) and the estimates (4.386) - (4.390), we obtain the energy inequality analogous to (3.285):

$$\begin{aligned}&\big(1-\tilde{C}_1t-\frac{\tilde{C}_1t}{\epsilon}\big)\big(\mathfrak{M}^{m+1}_p+\int_0^t\mathfrak{N}^{m+1}_p\big)+\big(1-\tilde{C}_2t-C_2t\big)\big(\hat{\mathfrak{M}}^{m+1}_p+\int_0^t\hat{\mathfrak{N}}^{m+1}_p\big)\\&\quad\le\mathfrak{M}_{\epsilon,p}(0)+\nu^l\int_0^t\mathfrak{N}^{m+1}_{p+1}+C\sum_{q=0}^{p-1}\int_0^t\mathfrak{N}^{m+1}_q+\mathcal{K}^p+\frac{\tilde{C}\lambda t}{\epsilon^4}\sup_{0\le s\le t}\mathcal{M}^{m+1}+C^*\int_0^t\mathcal{N}^{m+1},\end{aligned}\tag{4.391}$$

where ν is a small parameter and $\mathcal{K}^p$ is given by:

$$\mathcal{K}^p = C\big(t+\sqrt{\mathfrak{M}^m}+\lambda+\sup_{0\le s\le t}||f^m||_{L^2}\big)\int_0^t\big(\mathfrak{N}_p^m+\hat{\mathfrak{N}}_p^m\big)+\big(\frac{\tilde{C}t}{\lambda}+\frac{\tilde{C}t}{\lambda\epsilon^4}\big)\sup_{0\le s\le t}\mathfrak{M}_p^m.$$

Using the inequalities (4.384) and (4.391) in the same way as it is done in Subsection 3.2.6, we conclude the proof of Lemma 4.1. □

4.2 Local existence

The goal of this subsection is to prove the analogue of Theorem 3.4 stating the local existence, uniqueness and continuity of solutions to the regularized Stefan problem over a time interval that does *not* depend on ϵ. In addition to that, we state the fundamental energy estimate that will be used to establish the instability in Subsection 4.4.

Theorem 4.2 *Let $\zeta<0$. For any sufficiently small $L>0$ there exists $\tau>0$ depending on L and constants $0<\beta,\nu\le 1$ such that if*

$$M^\epsilon_{\beta,\nu}(u_0^\epsilon,f_0^\epsilon)\le\frac{L}{2}$$

then there exists a unique solution (u^ϵ,f^ϵ) to the regularized Stefan problem defined on the time interval $[0,\tau[$, satisfying the bound

$$\sup_{0\le t\le\tau}M^\epsilon_{\beta,\nu}(u^\epsilon,f^\epsilon f)(t)+\int_0^\tau N^\epsilon_{\beta,\nu}(u^\epsilon,f^\epsilon)(s)\,ds\le L$$

and $M^\epsilon_{\beta,\nu}(u^\epsilon,f^\epsilon)(\cdot)$ is continuous on $[0,\tau[$.

Proof. The proof of the local existence on an ϵ-dependent time interval $[0,\tau^\epsilon[$ together with the proof of continuity is fully analogous to the proofs in Subsections 3.3.1, 3.3.2 and 3.3.3. In order to prove that the solution exists on a short time interval independent of ϵ, we proceed analogously to the Subsection 3.3.5. We first pass to the limit as $m\to\infty$ in the inequality (4.391). All the terms involving $\tilde{C}$, $\bar{C}$ in the estimates vanish in the limit, because they are used to bound the cross-terms. Thus, in turn we obtain an estimate with constants that do not contain ϵ in the denominator:

$$\mathfrak{M}_p+\int_0^t\mathfrak{N}_p+\big(1-C_2t\big)\big(\hat{\mathfrak{M}}_p^\epsilon+\int_0^t\hat{\mathfrak{N}}_p^\epsilon\big)\le\delta_p+\nu^l\int_0^t\mathfrak{N}_{p+1}+C\sum_{q=0}^{p-1}\int_0^t\mathfrak{N}_q+\mathcal{K}_1+C^*\int_0^t\mathcal{N}+\frac{C}{\eta\lambda}\int_0^t||(u,f)||^2_{L^2},\tag{4.392}$$

where $||(u,f)||^2_{L^2}:=||u||^2_{L^2(\Omega)}+||f||^2_{L^2}$ and $\mathcal{K}_1=C\big(t+\sqrt{\mathfrak{M}}+\lambda+\sup_{0\le s\le t}||f||_{L^2}\big)\int_0^t\big(\mathfrak{N}_p+\hat{\mathfrak{N}}_p^\epsilon\big)$. Note that the key novelty with respect to the energy estimates in the stable case, is the presence of the (potentially) large multiple of the L^2-norm of (u,f) on RHS of (4.392). Multiplying (4.392) by ν^p and summing over $p=0,\dots,l-1$, just like in (3.355), we obtain

$$\begin{aligned}\sup_{0\le s\le t}\mathfrak{M}^\nu+\sum_{p=0}^{l-1}d_p\int_0^t\mathfrak{N}_p+(1-C_2t)\big(\sup_{0\le s\le t}\hat{\mathfrak{M}}^{\epsilon,\nu}+\int_0^t\hat{\mathfrak{N}}^{\epsilon,\nu}\big)\le\mathfrak{M}^\nu_\epsilon(0)+\epsilon\sum_{q=0}^{l-1}\nu^q\mathcal{K}_1+\sum_{q=0}^{l-1}\nu^qC^*\int_0^t\mathcal{N}\\+\frac{C\sum_{q=0}^{l-1}\nu^q}{\eta\lambda}\int_0^t||(u,f)||_{L^2},\end{aligned}\tag{4.393}$$

We now pass to the limit as $m\to\infty$ in (4.384). Note that the term $\frac{CC_B}{\lambda}(\xi_1+t\int_0^t\mathfrak{N}^{m+1})$ used to estimate $\int_0^t||f^{m+1}||^2_{L^2}$ can, in the limit, be replaced by the bound $\int_0^t||f||^2_{L^2}\le C\int_0^t||(u,f)||^2_{L^2}$. We obtain

$$\mathcal{M}_\epsilon+\int_0^t\mathcal{N}_\epsilon\le\mathcal{M}_\epsilon(0)+C\big(\lambda+\sup_{0\le s\le t}||f||_{H^4}+\sqrt{\mathfrak{M}}\big)\int_0^t\mathcal{N}_\epsilon+C\big(\lambda+\sqrt{\mathcal{M}}\big)\int_0^t\mathfrak{N}+C\int_0^t\mathcal{N}\sup_{0\le s\le t}\mathfrak{M}(s)+\frac{C}{\lambda}\int_0^t||(u,f)||^2_{L^2}.\tag{4.394}$$

Combining the estimates (4.393) and (4.394) in the same way as the estimates (3.355) and (3.356) are used in Subsection 3.3.5, we conclude that there exists a $\tau>0$ such that $\tau^\epsilon\ge\tau$ for any ϵ. □

4.3 Variational framework and linear growth

Let $\phi_0:\mathbb{S}^1\to S_1(x_0,0)$, $\phi_0(\theta)=(x_0+\cos\theta,\sin\theta)$ be the parametrization of $S_1(x_0,0)$. In order to find the growing mode λ_0 of the Stefan problem with surface tension, we turn our attention to the associated eigenvalue problem

$$\lambda v-\Delta v=0 \quad \text{on } \Omega\setminus S_1(x_0,0), \tag{4.395}$$

$$v\circ\phi_0=-f-f_{\theta\theta} \quad \text{on } \mathbb{S}^1, \tag{4.396}$$

$$[v_n]_-^+\circ\phi_0=-\lambda f \quad \text{on } \mathbb{S}^1, \tag{4.397}$$

$$v_n=0 \quad \text{on } \partial\Omega;\ [v]_-^+=0 \quad \text{on } S_1(x_0,0). \tag{4.398}$$

For any $k\in\mathbb{N}\cup\{0\}$ set

$$H^k(\Omega^\pm)=\Big\{u:\Omega\to\mathbb{R};\, u\mathbf{1}_{\mathrm{cl}(\Omega^\pm)}\in H^k(\mathrm{cl}(\Omega^\pm))\Big\}, \tag{4.399}$$

where $\Omega^+:=\Omega\setminus\mathrm{cl}(B_1(x_0,0))$ and $\Omega^-:=B_1(x_0,0)$. We define the linear operator $\mathcal{L}$

$$\mathcal{L}(v,f):=(\Delta v,-[\partial_n v]_-^+), \tag{4.400}$$

where the domain of $\mathcal{L}$ is given by

$$D(\mathcal{L}):=\Big\{(v,f)\in H^2(\Omega^\pm)\times H^{7/2}(\mathbb{S}^1);\ v_n=0 \text{ on } \partial\Omega;\ [v]_-^+=0 \text{ on } S_1(x_0,0);\ v\circ\phi_0=-f-f_{\theta\theta} \text{ on } \mathbb{S}^1\Big\}. \tag{4.401}$$

The following lemma states a number of important properties of the operator $\mathcal{L}$ and the proof can be found in [28]:

Lemma 4.3 *There are countably many eigenvalues $\lambda_0,\lambda_1,\lambda_2\dots$, they are all of finite multiplicity and the associated eigenvectors are smooth. There is exactly one simple positive eigenvalue λ_0. Furthermore, $\lambda_1=0$ and its eigenspace has dimension three. All other eigenvalues are negative.*

Let us define a bilinear operator $\langle\cdot,\cdot\rangle$ on $D(\mathcal{L})$. Let (v,f), $(w,g)\in D(\mathcal{L})$. Set

$$\langle(v,f),(w,g)\rangle:=\int_\Omega vw-\int_{\mathbb{S}^1}f(I+\partial_{\theta\theta})g=\int_\Omega vw+\int_{\mathbb{S}^1}(f_\theta g_\theta-fg)$$

Lemma 4.4 *The operator $\mathcal{L}$ is a symmetric operator on $D(\mathcal{L})$, with respect to $\langle\cdot,\cdot\rangle$. Furthermore, the eigenvectors associated to any two different eigenvalues are orthogonal with respect to $\langle\cdot,\cdot\rangle$.*

Proof. Where no confusion can arise, we abuse the notation and write $v=-f-f_{\theta\theta}$ on $\mathbb{S}^1$ instead of $v\circ\phi_0=-f-f_{\theta\theta}$. Analogous remark holds for $[v_n]_-^+=-\lambda f$ on $\mathbb{S}^1$. A straightforward computation yields:

$$\begin{aligned}\langle\mathcal{L}(v,f),(w,g)\rangle&=\big\langle(\Delta v,-[\partial_n v]_-^+),(w,g)\big\rangle=\int_\Omega\Delta v w+\int_{\mathbb{S}^1}[\partial_n v]_-^+(I+\partial_{\theta\theta})g\\&=\int_\Omega\Delta w v+\int_{\mathbb{S}^1}[\partial_n v]_-^+w-\int_{\mathbb{S}^1}[\partial_n w]_-^+v-\int_{\mathbb{S}^1}[v_n]_-^+\circ\phi_0 w=\int_\Omega\Delta w v-\int_{\mathbb{S}^1}[\partial_n w]_-^+v\\&=\int_\Omega\Delta w v-\int_{\mathbb{S}^1}-[\partial_n w]_-^+(I+\partial_{\theta\theta})f=\big\langle(v,f),(\Delta w,-[\partial_n w]_-^+)\big\rangle=\langle(v,f),\mathcal{L}(w,g)\rangle\,.\end{aligned}$$

Note that in the second equality, we used the Green's identity to rewrite $\int_\Omega\Delta v w$ and we also used the boundary condition $w=-(I+\partial_{\theta\theta})g$ to rewrite $\int_{\mathbb{S}^1}[\partial_n v]_-^+(I+\partial_{\theta\theta})g$. Note also that the third equality implies

$$\langle\mathcal{L}(v,f),(w,g)\rangle=-\int_\Omega\nabla v\cdot\nabla w. \tag{4.402}$$

If λ and μ are two different eigenvalues (w.l.o.g. $\mu\neq0$), with associated eigenvectors (v,f) and (w,g) respectively, then

$$\langle\mathcal{L}(v,f),(w,g)\rangle=\lambda\langle(v,f),(w,g)\rangle=\frac{\lambda}{\mu}\langle(v,f),\mathcal{L}(w,g)\rangle$$

and hence, due to the symmetry of $\langle\cdot,\cdot\rangle$, we have $\langle\mathcal{L}(v,f),(w,g)\rangle=0$. $\square$

To every eigenvalue λ_i with multiplicity $m(i)$, we associate the eigenvectors $e_{i,1},\dots,e_{i,m(i)}$, where we denote $e_{i,k}=(v_{i,k},z_{i,k})$ for $1\le k\le m(i)$. The null-space $\mathcal{S}$ of the operator $\mathcal{L}$ is spanned by three eigenvectors $e_{1,1}=(1,-1)$, $e_{1,2}=(0,\cos\theta)$ and $e_{1,3}=(0,\sin\theta)$ associated to the eigenvalue $\lambda_1=0$. Let us define the functional space D_1 by:

$$D_1:=\Big\{(v,f)\in H^1(\Omega^\pm)\times H^{5/2}(\mathbb{S}^1);\ v_n=0 \text{ on } \partial\Omega;\ [v]_-^+=0 \text{ on } S_1(x_0,0);\ v\circ\phi_0=-f-f_{\theta\theta} \text{ on } \mathbb{S}^1\Big\} \tag{4.403}$$

By $\mathcal{S}^\perp\subset D_1$ we denote the orthogonal complement of $\mathcal{S}$ in D_1 with respect to $\langle\cdot,\cdot\rangle$. In other words,

$$\mathcal{S}^\perp=\Big\{(v,f)\in D_1\big|\int_\Omega v+\int_{\mathbb{S}^1}f=0,\ \mathbb{P}_1 f=0\Big\}.$$

Lemma 4.5 *The following variational characterization of the positive eigenvalue λ_0 holds:*

$$-\frac{1}{\lambda_0}=\min_{(v,f)\in\mathcal{S}^\perp}\frac{\langle(v,f),(v,f)\rangle}{\int_\Omega|\nabla v|^2}.$$

Proof. Without loss of generality, we assume $x_0=0$. For $(v,f)\in\mathcal{S}^\perp$ let us set

$$\mathcal{I}(v,f):=\frac{\langle(v,f),(v,f)\rangle}{\int_\Omega|\nabla v|^2}. \tag{4.404}$$

Let $\iota=\inf_{(v,f)\in\mathcal{S}^\perp}\mathcal{I}(v,f)$.

Boundedness from below: let $(v,f)\in\mathcal{S}^\perp$. Since $\int_\Omega v+\int_{\mathbb{S}^1}f=0$ and $\int_{\mathbb{S}^1}v=-\int_{\mathbb{S}^1}f$, we deduce that $\int_\Omega v=\int_{\mathbb{S}^1}v$. By the mean value theorem $\left|\frac{1}{|\Omega|}\int_\Omega v-\frac{1}{|\mathbb{S}^1|}\int_{\mathbb{S}^1}v\right|\le C||\nabla v||_{L^2(\Omega^\pm)}$, and thus $|\zeta|\left|\int_\Omega v\right|\le C||\nabla v||_{L^2(\Omega^\pm)}$, where $||\cdot||_{L^2(\Omega^\pm)}$ is defined by (4.399). Since $|\zeta|>0$, we conclude

$$\langle(v,f),(v,f)\rangle=\int_\Omega|\mathbf{P}v|^2+\int_{\mathbb{S}^1}(f_\theta^2-|\mathbb{P}f|^2)+\zeta\Big(\int_\Omega v\Big)^2\ge\zeta\Big(\int_\Omega v\Big)^2\ge-\frac{C}{\zeta}||\nabla v||^2_{L^2(\Omega^\pm)},$$

and hence $\iota>-\infty$.

Negativity: $\iota<0$. Assume for the moment $x_0=0$, i.e. $S_1(x_0,0)=\mathbb{S}^1$. Let

$$v_{d,\epsilon}(r)=\begin{cases} d \quad 0\le r\le 1-\epsilon \text{ and } 1+\epsilon\le r\le R_*,\\ d+\frac{1-d}{\epsilon}(r-(1-\epsilon)), \quad 1-\epsilon\le r\le 1,\\ v_{d,\epsilon}(2-r), \quad 1\le r\le 1+\epsilon.\end{cases}$$

Note that $v_{d,\epsilon}(1)=1$ and we define $f_{d,\epsilon}=-1$. Hence $\mathbb{P}_1 f_{d,\epsilon}=0$ for any d,ϵ. For a fixed d note that $\int_\Omega v_{d,\epsilon}=R_*^2 d\pi+O(\epsilon)$ and thus the condition $\int_\Omega v_{d,\epsilon}+\int_{\mathbb{S}^1}f_{d,\epsilon}=0$ reads $R_*^2 d\pi-2\pi=O(\epsilon)$ and hence $d=2/R_*^2+O(\epsilon)$. On the other hand, it is easy to see that $\int_\Omega|\mathbf{P}v_{d,\epsilon}|^2=O(\epsilon)$. Note however

$$\mathcal{I}(v_{d,\epsilon},f_{d,\epsilon})=\zeta\Big(\int_\Omega v_{d,\epsilon}\Big)^2+\int_\Omega|\mathbf{P}v_{d,\epsilon}|^2=\zeta R_*^4d^2\pi^2+O(\epsilon)<0$$

for small ϵ. To conclude that $\iota<0$, we slightly perturb the function $v_{d,\epsilon}$ to a smooth function v_{neg} so that the $\int_\Omega|\nabla v_{\text{neg}}|^2$ is well defined. (If $x_0\neq0$, we define the function $v_{\text{neg}}^{x_0}$ inside $S_{1+\epsilon}(x_0,0)$ by simply translating the function v_{neg} by x_0. On $B_{R_*}(0)\setminus S_{1+\epsilon}(x_0,0)$ set $v_{\text{neg}}^{x_0}=d$. It is easy to see that $\mathcal{I}(v_{\text{neg}}^{x_0},-1)<0$.)

Existence of the minimizer. Let (v_n,f_n) be an infimizing sequence for the above variational problem. Without loss of generality assume $\int_\Omega|\nabla v_n|^2=1$ (otherwise rescale). As in the proof of boundedness,

$$\Big|\int_\Omega v_n\Big|^2\le\frac{C}{\zeta}||\nabla v_n||_{L^2(\Omega^\pm)} \tag{4.405}$$

and hence from here and the Poincaré inequality $||v_n||_{L^2(\Omega^\pm)}\le C||\nabla v_n||_{L^2(\Omega^\pm)}=C$. Due to (4.405), $\mathbb{P}_1 f_n=0$ and the condition $v|\mathbb{S}^1=-f-f_{\theta\theta}$, we deduce that there exists a positive constant C such that $||f_n||_{H^2}\le C$

for any n. By Banach-Alaoglu, we deduce that there exist weak limits v^* and f^* of $\{v_n\}$ and $\{f_n\}$ in $H^1(\Omega)$ and $H^2(\mathbb{S}^1)$ respectively. Hence $\langle(v_n,f_n),(v_n,f_n)\rangle \to \langle(v^*,f^*),(v^*,f^*)\rangle$ as $n\to\infty$. Furthermore, by weak lower semi-continuity of $||\cdot||_{H^1}$, we deduce $1=\liminf_{n\to\infty}||\nabla v_n||_{L^2(\Omega^\pm)} \geq ||\nabla v^*||_{L^2(\Omega^\pm)}$. Since $\mathcal{I}(v_n,f_n)<0$ for n large enough, we deduce

$$\iota = \liminf_{n\to\infty} \mathcal{I}(v_n,f_n) \geq \liminf_{n\to\infty} \frac{\langle(v_n,f_n),(v_n,f_n)\rangle}{||\nabla v^*||^2_{L^2(\Omega^\pm)}} = \mathcal{I}(v^*,f^*)$$

and hence (v^*,f^*) is a minimizer. Note that $||\nabla v^*||_{L^2(\Omega^\pm)}=1$. Otherwise assume $||\nabla v^*||_{L^2(\Omega^\pm)}<1$. Then $\langle(v^*,f^*),(v^*,f^*)\rangle = \iota||\nabla v^*||^2_{L^2(\Omega^\pm)} > \iota = \liminf_{n\to\infty}\langle(v_n,f_n),(v_n,f_n)\rangle = \langle(v^*,f^*),(v^*,f^*)\rangle$, which is a contradiction (note that we used the fact $\iota<0$).
Euler-Lagrange equation. Setting $\iota=-\frac{1}{\lambda_0}$, the Euler-Lagrange equation for the variational problem of minimizing $\mathcal{I}$ over $\mathcal{S}^\perp$ allows to conclude that the minimizer (v^*,f^*) is the weak solution of the eigenvalue problem $\mathcal{L}(v,f)=\lambda_0(v,f)$, $(v,f)\in D_1$. In other words, for any test function $\varphi\in C^\infty(\Omega^\pm)$:

$$\int_\Omega \big\{\lambda_0 v^*\varphi - \nabla v^*\cdot\nabla\varphi\big\} - \lambda_0\int_{\mathbb{S}^1} f\varphi = 0. \tag{4.406}$$

By standard elliptic regularity, it is easy to see that v^* is smooth in the interior of Ω^+ and Ω^-. We want to prove that v^* is H^2 up to the boundary i.e. $v^*\in H^2(\Omega^\pm)$ (recall (4.399)).

Claim. Let $\gamma_1\in H^1(\Omega^\pm)$ and $\gamma_2\in H^2(\mathbb{S}^1)$ be given. Let $v\in H^1(\Omega^\pm)$ be the weak solution of the problem

$$\Delta v = \gamma_1 \quad \text{in } \Omega,$$

$$[\partial_n v]^+_- = \gamma_2 \quad \text{on } \mathbb{S}^1,$$

$$\partial_n v = 0 \quad \text{on } \partial\Omega.$$

Then $v\in H^2(\Omega^\pm)$.

Proof of the Claim. Written in polar coordinates, this means that for any test function $\varphi\in C^\infty(\Omega^\pm)$, the following holds:

$$\int_0^{R^*}\int_{\mathbb{S}^1}\big\{-v_r\varphi_r r - \frac{v_\theta\varphi_\theta}{r}\,dr\,d\theta\big\} - \int_{\mathbb{S}^1}\gamma_2\varphi\,d\theta = \int_0^{R^*}\int_{\mathbb{S}^1}\gamma_1\varphi\,dr\,d\theta. \tag{4.407}$$

Let $\nabla^h_\theta v(r,\theta)=\frac{v(r,\theta+h)-v(r,\theta)}{h}$ for $r\neq 0$. Let $\varphi\in C^\infty(\Omega)$ be supported on the complement of a small ball $B_\delta(0)$. From (4.407), we immediately obtain

$$\int_\delta^{R^*}\int_{\mathbb{S}^1}\big\{-\nabla^h_\theta v_r\varphi_r r - \frac{\nabla^h_\theta v_\theta\varphi_\theta}{r}\,dr\,d\theta\big\} - \int_{\mathbb{S}^1}\nabla^h_\theta\gamma_2\varphi\,d\theta = \int_\delta^{R^*}\int_{\mathbb{S}^1}\nabla^h_\theta\gamma_1\varphi r\,dr\,d\theta.$$

Let $s>0$ be a fixed small real number. Let $\varphi\in H^2(\Omega^\pm)$ be a test function such that $\varphi(r,\theta)=-\nabla^h_\theta v$ for $(r,\theta)\in O_1:=[1-s,1+s]\times\mathbb{S}^1$ and $\text{supp}(\varphi)\in[\delta,R_*]\times\mathbb{S}^1$. Let us also denote $O_2:=\big\{[\delta,1-s]\cup[1+s,R_*]\big\}\times\mathbb{S}^1$. Then

$$\begin{aligned}\int_{O_1}\big\{|\nabla^h_\theta v_r|^2 r + \frac{|\nabla^h_\theta v_\theta|^2}{r}\big\}\,dr\,d\theta &= -\int_{\mathbb{S}^1}\nabla^h_\theta\gamma_2\nabla^h_\theta v\,d\theta \\ -\int_{O_1\cup O_2}\nabla^h_\theta\gamma_1\nabla^h_\theta\varphi r\,dr\,d\theta &+ \int_{O_2}\big\{\nabla^h_\theta v_r\varphi_r r + \frac{\nabla^h_\theta v_\theta\varphi_\theta}{r}\,dr\,d\theta\big\}\end{aligned} \tag{4.408}$$

Note that

$$\Big|\int_{\mathbb{S}^1}\nabla^h_\theta\gamma_2\nabla^h_\theta v\,d\theta\Big| = \Big|\int_{\mathbb{S}^1}\nabla^{-h}_\theta\nabla^h_\theta\gamma_2 v\,d\theta\Big| \leq ||\gamma_2||_{H^2}||v||_{L^2(\mathbb{S}^1)} \leq ||\gamma_2||_{H^2}||v||_{H^1(\Omega^\pm)},$$

where we used the trace inequality in the last estimate. Furthermore,

$$\int_{O_1\cup O_2}\nabla^h_\theta\gamma_1\nabla^h_\theta\varphi r\,dr\,d\theta \leq ||\nabla^h_\theta\gamma_1||_{L^2(O_1\cup O_2)}||\nabla^h_\theta\phi||_{L^2(O_1\cup O_2)} \leq C||\gamma_1||_{H^1(\Omega^\pm)}\big(||v||_{H^1(\Omega^\pm)} + ||\varphi||_{H^1(O_2)}\big).$$

Using the interior regularity, it easy to bound the remaining term on RHS of (4.408):

$$\Big|\int_{O_2}\big\{\nabla_\theta^h v_r\varphi_r r+\frac{\nabla_\theta^h v_\theta\varphi_\theta}{r}\,dr\,d\theta\big\}\Big|\leq C||v||_{H^2(O_2)}||\varphi||_{H^1(O_2)}\leq C||v||_{H^1(\Omega^\pm)}||\varphi||_{H^1(\Omega^\pm)}$$

Hence, we may pass to the limit as $h\to\infty$ in (4.408) to conclude $v_r,v_\theta\in H_\theta^1(\Omega^\pm)$. Since $\Delta v^+=(\partial_{rr}+\frac{\partial_r}{r}+\frac{\partial_{\theta\theta}}{r^2})v^+$, we deduce that

$$\partial_{rr}v^+=\gamma_1\mathbf{1}_{\mathrm{cl}(\Omega^+)}-(\frac{\partial_r}{r}+\frac{\partial_{\theta\theta}}{r^2})v^+\qquad\text{in distributional sense.}$$

Since the RHS of the above expression actually belongs to $L^2(\mathrm{cl}(\Omega^+))$, we deduce $v^+_{rr}\in L^2(\mathrm{cl}(\Omega^+))$ in distributional sense and thus v^+ is H^2 up to the boundary $\mathbb{S}^1$. We argue similarly for v^- to conclude the proof of the Claim.

We now apply the above Claim to the function $v^*\in H^1(\Omega^\pm)$. Upon setting $\gamma_1=\lambda_0v^*$ and $\gamma_2=-\lambda_0f^*$, we conclude $v^*\in H^2(\Omega^\pm)$. Hence λ_0 is a positive eigenvalue for the linearized Stefan problem, with an associated eigenvector (v^*,f^*). Note that $v^*|_{\mathbb{S}^1}\in H^{3/2}(\mathbb{S}^1)$ and the Dirichlet boundary condition $v^*|_{\mathbb{S}^1}=-f^*-f^*_{\theta\theta}$ implies $f^*\in H^{7/2}(\mathbb{S}^1)$. By a classical bootstrap argument, we conclude that $(v^*,f^*)\in C^\infty(\Omega^\pm)\times C^\infty(\mathbb{S}^1)$. □

Lemma 4.6 *Let $e_{0,1}=(v_{0,1},z_{0,1})$ be an eigenvector associated to the growing mode λ_0, for the Stefan problem linearized around $S_1(0,0)=\mathbb{S}^1$. Then $v_{0,1}$ is spherically symmetric.*

Proof. Let $v_{0,1}(r,\theta)=\sum_{k=0}^\infty P_k(r,\theta)$ be the Fourier decomposition of $v_{0,1}$. Due to the linear independence of $\big\{\cos k\theta,\sin k\theta\big\}_k$, we deduce that each P_k solves the eigenvalue problem (1.15) - (1.18) with $\lambda=\lambda_0>0$. If $k\geq1$, it is easy to see that $\langle(P_k,P_k(1,\theta)),(P_k,P_k(1,\theta))\rangle=\int_\Omega P_k^2+\int_{\mathbb{S}^1}(\partial_\theta P_k(1,\theta))^2-(P_k(1,\theta))^2\geq0$. On the other hand, using (4.402) and the previous observation

$$-\int_\Omega|\nabla P_k|^2=\langle\mathcal{L}(P_k,P_k(1,\theta)),(P_k,P_k(1,\theta))\rangle=\lambda_0\langle(P_k,P_k(1,\theta)),(P_k,P_k(1,\theta))\rangle\geq0.$$

This is only possible if $P_k=0$ for all $k\geq1$. From here we immediately deduce that $(v_{0,1},z_{0,1})=(P_0(r),c)$ for some function P_0 and a constant $c=-P_0(1)$. □

Now, we shall turn $\mathcal{S}^\perp$ into a normed space, by introducing a new norm $||\cdot||_I$ for any fixed $I>0$. The new scalar product $\langle\cdot,\cdot\rangle_I$ is defined by:

$$\langle(v,f),(w,g)\rangle_I=\langle(v,f),(w,g)\rangle+(\frac{1}{\lambda_0}+I)\int_\Omega\nabla v\cdot\nabla w,\quad(v,f),\,(w,g)\in\mathcal{S}^\perp.\tag{4.409}$$

From Lemma 4.5, we immedaitely see that $\langle(v,f),(v,f)\rangle_I\geq I||\nabla v||^2_{L^2(\Omega^\pm)}\geq0$. Furthermore, if $\langle(v,f),(v,f)\rangle_I=0$, then $||\nabla v||_{L^2(\Omega^\pm)}=0$ and hence $v\equiv$ const, which implies $(v,f)=k_1(1,-1)+k_2(0,\cos\theta)+k_3(0,\sin\theta)=k_1e_{1,1}+k_2e_{1,2}+k_3e_{1,3}$, for some reals k_1,k_2,k_3. Since $(v,f)\in\mathcal{S}^\perp$, it follows $(v,f)=(0,0)$. Thus, $\langle\cdot,\cdot\rangle_I$ is an inner product on $\mathcal{S}^\perp$ and it defines the norm $||\cdot||_I:=\sqrt{\langle\cdot,\cdot\rangle_I}$.

Lemma 4.7 *There exists a constant $C>0$ such that for any $(v,f)\in\mathcal{S}^\perp$, $||v||_{L^2(\Omega)}+||f||_{L^2}\leq C||(v,f)||_I$.*

Proof. As in the proof of Lemma 4.4, we abuse notation and write $v\circ\phi_0=v$. Let $(v,f)\in\mathcal{S}^\perp$. As in the proof of boundedness in the previous lemma, we conclude $|\zeta|\big|\int_\Omega v\big|\leq C||\nabla v||_{L^2(\Omega^\pm)}$. From this and the Poincaré inequality, we deduce $||v||_{L^2(\Omega)}\leq C||\nabla v||_{L^2(\Omega^\pm)}$. Furthermore, from $v|\mathbb{S}^1=-f-f_{\theta\theta}$, we obtain $\int_{\mathbb{S}^1}f_\theta^2-|\mathbb{P}f|^2\leq C||v||_{L^2(\mathbb{S}^1)}+C|\int_{\mathbb{S}^1}f|^2$. Since $\mathbb{P}_1f=0$ and $|\int_{\mathbb{S}^1}f|=|\int_{\mathbb{S}^1}v|\leq C||v||_{L^2(\mathbb{S}^1)}$, from the trace inequality we deduce

$$||f||^2_{L^2}=\int_{\mathbb{S}^1}f_\theta^2-|\mathbb{P}f|^2+\frac{1}{|\mathbb{S}^1|}\Big(\int_{\mathbb{S}^1}f\Big)^2\leq C||v||^2_{L^2(\mathbb{S}^1)}\leq C||v||^2_{H^1(\Omega^\pm)}\leq C||\nabla v||^2_{L^2(\Omega^\pm)}.$$

Since $||(v,f)||_I^2\geq I||\nabla v||^2_{L^2(\Omega^\pm)}$, we conclude the claim of the lemma. □

Lemma 4.8 *The set* $\mathcal{B}=\cup_{i=0}^{\infty}\cup_{k\le m(i)}\{e_{i,k}\}$ *forms an orthonormal basis of* $\mathcal{S}^{\perp}$ *with respect to* $\langle\cdot,\cdot\rangle_I$.

Proof. Without loss of generality, we assume $x_0=0$. Let λ and λ^* be two different eigenvalues with associated eigenvectors $e=(v,z)$ and $e^*=(v^*,z^*)$. Lemma 4.4 implies $\langle e,e^*\rangle=0$. From (4.402), we deduce $\int_\Omega \nabla v\cdot\nabla v^*=0$ and hence $\langle e,e^*\rangle_I=0$. Let us assume span$[\mathcal{B}]\neq\mathcal{S}^\perp$ and let $\mathcal{B}^\perp$ denote the orthogonal complement of span$[\mathcal{B}]$ with respect to $\langle\cdot,\cdot\rangle_I$. Let us set

$$J:=\inf_{(v,f)\in\mathcal{B}^\perp}\mathcal{I}(v,f),$$

where $\mathcal{I}$ is defined by (4.404). We distinguish two cases: $J<0$ and $J>0$.
Case 1: $J<0$. Note that $J\ge\iota$, where ι is defined in the line after (4.404). The same proof as in Lemma 4.5 yields the existence of a smooth minimizer $(\bar v,\bar f)\in\mathcal{B}^\perp$ such that $\mathcal{I}(\bar v,\bar f)=J$. From the symmetry of $\mathcal{L}$, we deduce

$$\left\langle\mathcal{L}(\bar v,\bar f),(v_{i,k},f_{i,k})\right\rangle_I=\left\langle(\bar v,\bar f),\mathcal{L}(v_{i,k},f_{i,k})\right\rangle_I=\left\langle(\bar v,\bar f),\lambda_i(v_{i,k},f_{i,k})\right\rangle_I=0,$$

since $(\bar v,\bar f)\in\mathcal{B}^\perp$. Hence $\mathcal{L}(\bar v,\bar f)\in\mathcal{B}^\perp$ and if we set $J=-\frac{1}{\bar\lambda}$, the Euler-Lagrange variational equation implies that $\bar\lambda$ is an eigenvalue of $\mathcal{L}$ with an associated eigenvector $(\bar v,\bar f)\in\mathcal{B}^\perp$, which is a contradiction.
Case 2: $J>0$. In this case, we analyze the problem of minimizing

$$\frac{1}{\mathcal{I}(v,f)}=\frac{\int_\Omega|\nabla v|^2}{\langle(v,f),(v,f)\rangle}$$

over the set $\mathcal{B}^\perp$. Since $J>0$, it is clear that $\frac{1}{\mathcal{I}(v,f)}>0$ for all $(v,f)\in\mathcal{B}^\perp$. Set $K=\inf_{(v,f)\in\mathcal{B}^\perp}\frac{1}{\mathcal{I}(v,f)}$ and let (v_n,f_n) be an infimizing sequence. After rescaling assume $\langle(v_n,f_n),(v_n,f_n)\rangle=1$. Just like in the proof of Lemma 4.5, we conclude that there exists a $C>0$ independent of n such that for any $n\in\mathbb{N}$ $||v_n||_{H^1(\Omega^\pm)}+||f_n||_{H^2}\le C$. We may pass to a weak limit $(\bar v,\bar f)$ by Banach-Alaoglu. From the compactness of the embedding $H^1\hookrightarrow L^2$ and $H^2\hookrightarrow H^1$, we conclude $\langle(v_n,f_n),(v_n,f_n)\rangle\to\left\langle(\bar v,\bar f),(\bar v,\bar f)\right\rangle$ as $n\to\infty$. Furthermore, by weak lower semi-continuity, $\liminf_{n\to\infty}||\nabla v_n||_{L^2(\Omega^\pm)}\ge||\nabla\bar v||_{L^2(\Omega^\pm)}$. Thus

$$K=\liminf_{n\to\infty}\frac{1}{\mathcal{I}(v_n,f_n)}\ge\frac{1}{\mathcal{I}(\bar v,\bar f)},$$

and thus $(\bar v,\bar f)$ is the minimizer. If we set $K=-\bar\lambda$, the Euler-Lagrange variational equation implies that $\bar\lambda$ is an eigenvalue of $\mathcal{L}$ with an associated eigenvector $(\bar v,\bar f)\in\mathcal{B}^\perp$, which is a contradiction. This finishes the proof of the lemma. □

For a given $y\in D_1$, let $y=y_{\mathcal{S}}+y^\perp$, where $y_{\mathcal{S}}$ is the projection onto $\mathcal{S}$ and $y^\perp$ projection onto $\mathcal{S}^\perp$, with respect to $\langle\cdot,\cdot\rangle$. We define

$$||y||^2:=||y_{\mathcal{S}}||^2_{L^2(\Omega)\times L^2(\mathbb{S}^1)}+||y^\perp||_I^2. \tag{4.410}$$

Due to Lemma 4.7, we immediately see that

$$||y||_{L^2(\Omega)\times L^2(\mathbb{S}^1)}\le C||y||. \tag{4.411}$$

Lemma 4.9 *Let* $(w_0,f_0)=\sum_{i=0}^{\infty}\sum_{1\le k\le m(i)}a_{i,k}e_{i,k}$. *Then there exists a constant* $C_{\mathcal{L}}$, *such that*

$$||e^{\mathcal{L}t}(w_0,f_0)||\le C_{\mathcal{L}}e^{\lambda_0 t}||(w_0,f_0)||.$$

Proof. Note that

$$\begin{aligned}
||e^{\mathcal{L}t}(w_0,f_0)||^2 &= ||\sum_{1\le k\le3}a_{1,k}e_{1,k}+\sum_{i\ge0,i\neq1}e^{\lambda_i t}\sum_{1\le k\le m(i)}a_{i_k}e_{i,k}||^2\\
&=||\sum_{1\le k\le3}a_{1,k}e_{1,k}||^2_{L^2(\Omega)\times L^2(\mathbb{S}^1)}+e^{2\lambda_0 t}|a_{0,1}|^2+\sum_{i\ge2}e^{2\lambda_i t}\sum_{1\le k\le m(i)}|a_{i,k}|^2\\
&\le Ce^{2\lambda_0 t}\sum_{i,k}|a_{i,k}|^2,
\end{aligned}$$

and the claim follows. □

4.4 Proof of Theorem 1.3

The abstract formulation. We shall use the bootstrap method developed in [19] to prove the instability of the stationary surfaces in the case $\zeta<0$. The following lemma is a simple modification of Lemma 1 in [18], which suffices to establish the *nonlinear* instability of steady spheres for the Stefan problem.

Lemma 4.10 *Assume that X is a Banach space with the associated norm $||\cdot||_X$ and $\mathcal{L}:D\subset X\to X$ is a linear operator. Assume a norm $||\cdot||$ with the property $||\cdot||_X\le C||\cdot||$ and such that $e^{t\mathcal{L}}$ generates a strongly continuous semigroup satisfying*

$$||e^{t\mathcal{L}}||_{X\to X}\le C_{\mathcal{L}}e^{\lambda_0 t} \tag{4.412}$$

for some $C_{\mathcal{L}}$ and $\lambda_0>0$ ($||\cdot||_{X\to X}$ stands for the operator norm). Assume a nonlinear operator $U:D\to X$, a norm $|||\cdot|||$ on D and a constant C_U, such that

$$||U(y)||\le C_U|||y|||^2 \tag{4.413}$$

for all $y\in D$ and $|||y|||<\infty$. Assume that there exists a small constant σ such that for any solution $y(t)$ to the equation $y'=\mathcal{L}y+U(y)$, with $|||y(t)|||\le\sigma$, there exist constants C_1 and C_2 and a small constant μ, such that the following energy estimate holds:

$$|||y(t)|||^2\le C_1|||y_0|||^2+C_2\int_0^t||y(s)||^2\,ds. \tag{4.414}$$

Consider a family of initial data $y^\delta(0)=\delta y_0$ with $||y_0||=1$ and $|||y_0|||<\infty$ and let θ_0 be a sufficiently small (fixed) number. Then there exists some constant $C>0$ such that if

$$0\le t\le T:=\frac{1}{\lambda}\log\frac{\theta_0}{\delta},$$

we have

$$\big|\big|y(t)-\delta e^{\mathcal{L}t}y_0\big|\big|\le C\Big(|||y_0|||^2+1\Big)\delta^2e^{2\lambda_0t}.$$

In particular, if there exists a constant C_p such that $||\delta e^{\mathcal{L}t}y_0||\ge C_p\delta e^{\lambda_0 t}$, then there exists en escape time $T^\delta\le T$ such that

$$\big|\big|y(T^\delta)\big|\big|\ge K_0>0, \tag{4.415}$$

where K_0 depends explicitly on $C_{\mathcal{L}}$, C_U, C_2, C_p, y_0, λ_0 and is independent of δ.

Proof of nonlinear instability. Let $M^{**}\le\frac{L}{2}$, where L is given by Theorem 4.2 and let (u,f) be the unique solution of the regularized Stefan problem with the initial data satisfying $E^\epsilon_{\beta,\nu}(u^\epsilon_0,f^\epsilon_0)\le M^{**}$. We multiply (4.393) by β and sum it with (4.394). In analogy to (3.357) and (3.358), we choose λ, M^{**}, ν small and use the smallness of $||f||_{H^l}$ to arrive at

$$\begin{aligned}\sup_{0\le s\le\tau}M_{\beta,\nu}(s)+\frac12\int_0^\tau N_{\beta,\nu}(s)\,ds+(1-Ct)\sup_{0\le s\le\tau}\hat{M}_{\beta,\nu}(s)+\frac12\int_0^\tau\hat{N}_{\beta,\nu}(s)\,ds\\ \le M_{\beta,\nu}(0)+\hat{M}_{\beta,\nu}(0)+\epsilon\mathcal{K}_2+C\int_0^\tau||(u,f)||^2_{L^2}\,ds,\end{aligned}$$

where the constant $\mathcal{K}_2$ is defined analogously to $\mathcal{J}_2$ as in the line after (3.357). Passing to the limit as $\epsilon\to0$, we obtain a solution (u,f) to the original Stefan problem. Note that $\hat{M}_{\beta,\nu}(0)\to0$ as $\epsilon\to0$, by the choice of the initial condition. Also, $\epsilon\mathcal{K}_2\to0$ as $\epsilon\to0$ because $\mathcal{K}_2$ is bounded. As a result, we obtain the estimate

$$\sup_{0\le s\le\tau}M_{\beta,\nu}(u,f)(s)+\frac12\int_0^\tau N_{\beta,\nu}(u,f)(s)\,ds\le M_{\beta,\nu}(u_0,f_0)+C\int_0^\tau||(u,f)||^2_{L^2}\,ds \tag{4.416}$$

as stated in Theorem 1.3. To apply Lemma 4.10 and finish the proof of Theorem 1.3, we have to formulate the Stefan problem on a fixed domain. To this end we apply the change of variables described in Subsection 3.3.1: we set $\bar{x}=\pi(t,x)x$, where π is defined by dropping the super-indices in the definition (3.289). Without loss

of generality, we furthermore assume $x_0=0$. Hence, setting $v:=u\circ\pi$ leads to an equation for the function $v:\Omega_{\mathbb{S}^1}\to\mathbb{R}$:

$$v_t-\Delta v=a_{ij}u_{x^ix^j}+b_iu_{x^i}, \tag{4.417}$$

where a_{ij} and b_i are defined by dropping the index m in (3.291). From (3.292), we deduce

$$[v_n]_-^+=-\frac{f_tR^3}{|g|^2}\text{ on }\mathbb{S}^1. \tag{4.418}$$

The Dirichlet boundary condition simply reads:

$$v=-f-f_{\theta\theta}+N(f)\text{ on }\mathbb{S}^1. \tag{4.419}$$

In order to set the problem (4.417) - (4.419) in the framework of Lemma 4.10, we have to consider the new unknown $w(x):=v(x)-N(f(\frac{x}{|x|}))$. This way we insure that the nonlinear boundary condition (4.419) transforms into a linear condition for w. We obtain

$$w_t-\Delta w=\mathcal{R}(w,f),\quad \text{in }\Omega_{\mathbb{S}^1}, \tag{4.420}$$

$$w=-f-f_{\theta\theta}\quad\text{on }\mathbb{S}^1, \tag{4.421}$$

$$[w_n]_-^+=-\frac{f_tR^3}{|g|^2}\text{ on }\mathbb{S}^1, \tag{4.422}$$

$$w_n=0\quad\text{on }\partial\Omega. \tag{4.423}$$

Here

$$\begin{aligned}\mathcal{R}(w,f)&=a_{ij}v_{x^ix^j}+b_iv_{x^i}-(\partial_t-\Delta)N(f(\frac{\cdot}{|\cdot|}))\\&=a_{ij}\big(w+N(f(\frac{\cdot}{|\cdot|}))\big)_{x^ix^j}+b_i\big(w+N(f(\frac{\cdot}{|\cdot|}))\big)_{x^i}-(\partial_t-\Delta)N(f(\frac{\cdot}{|\cdot|})).\end{aligned}$$

To apply Lemma 4.10 set $X:=L^2(\Omega)\times L^2(\mathbb{S}^1)$ and $D=D_1$ as defined in (4.403). The norm $||\cdot||$ is defined by (4.410) and we also define

$$|||(w,f)|||^2:=||w||^2_{H^{2l-1}(\Omega^\pm)}+||f_\theta||^2_{H^{2l-1}}$$

Observe that if $(w,f)\in X$ is a solution to the Stefan problem (4.420) - (4.423), then the norm $|||\cdot|||$ is equivavlent to the norms defined by $M_{\beta,\nu}$. This follows easily from the boundedness of the Jacobian of the coordinate transformation $\bar{x}=\pi(t,x)x$ and the fact that the time derivatives of w and f can be recursively expressed in terms of purely spatial derivatives via the relations $w_t=\Delta w+F$ and $f_t=-\frac{|g|^2}{R^3}[w_n]_-^+$. From this remark, the bound (4.411) and the inequality (4.416), we immediately conclude that there exist constants C_1 and C_2, such that the inequality (4.414) is satisfied. Furthermore, we define

$$U(w,f)=(\mathcal{R}(w,f),-(\frac{|g|^2}{R^3}-1)[w_n]_-^+). \tag{4.424}$$

Note that the time derivative of f that appears in the definition of $\mathcal{R}(w,f)$ can be alternatively expressed as a purely spatial differential operator via (4.422). With this notation, the problem (4.420) - (4.423) takes the form $\partial_t(w,f)=\mathcal{L}(w,f)+U(w,f)$. It is straightforward to verify $||U(w,f)||\le C|||(w,f)|||^2$ and thus, the assumption (4.413) from Lemma 4.10 is satisfied whereas Lemma 4.9 guarantees (4.412). We may apply Lemma 4.10 to conclude that (1.41) holds and this finishes the proof of Theorem 1.3. □

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